CS468 Notes

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1 Review

$\gamma(t)$ gives us the trace of the curve. Think of t as standing for time.
The arc length $s(t)$ can be found via

$$\int_0^t \|\gamma'\| dt \tag{1}$$

This tells how far you've traveled at time t.
So, now we can take $\gamma(s)$. This gives us the trace of the curve.
$\dot{\gamma}(s)$ is automatically unit length if parameterized by arc length. The reason is that we basically removed the acceleration, so we only go unit per second.

2 Frenet Frame

In 3-space, we can associate 3 orthogonal vectors with γ at each point.

2.1 Tangent

The first orthogonal vector is the unit tangent vector. The other two vectors are the normal and the binormal. This first order of the tangent is a good approximation of the curve.

2.2 Normal

We can get the normal by taking the derivative of the tangent and dividing it by is length. The normal is sort of the plane that your curve sits in.

2.3 Binormal

In order to find the binormal, we just take cross product of the normal and tangent, because we know they are orthogonal. The binormal is just some third thing.

$$\frac{dT}{ds} = \kappa N \tag{2}$$

$$\frac{dN}{ds} = -\kappa T + \tau B \tag{3}$$

$$\frac{dB}{ds} = -\tau N \tag{4}$$

κ describes how much you are turning. The length of the derivative of the tangent. τ is torsion that describes how much the curve is leaving the plane that approximates it

2.4 Potential Discretization

Discrete Frenet Frames have been used to understand the joints between molecules on proteins. Procedure:

1. Compute the tangent of a piecewise linear curve. Important note is that the unit tangent will be associated with edges and not with points.

$$t_j = \frac{p_{j+1} - p_j}{\|p_{j+1} - p_j\|} \tag{5}$$

2. The binormal is the normal to the plane that the normal and the tangent span. So we can get that via

$$b_j = t_{j-1} \times t_j \tag{6}$$

3. The normal is orthogonal to the binormal and the tangent so we can get it via

$$n_j = b_j \times t_j \tag{7}$$

4. We then look at the rotation and torsion of the rotation matrices.

$$t_k = R(b_k, \theta_k) t_{k-1}$$
$$b_{k+1} = R(t_k, \phi_k) b_k$$

Calculate T, B, and N as orthogonal vectors.
Bond and torsion angles (derivatives converge to κ and τ, resp.)

2.5 Transfer Matrix

$$\begin{pmatrix} t_{i+1} \\ n_{i+1} \\ b_{i+1} \end{pmatrix} = R_{i+1,i} \begin{pmatrix} t_i \\ n_i \\ b_i \end{pmatrix} \tag{8}$$

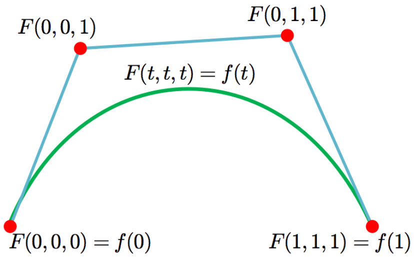
This contraction carries us from one tangent to another. Work has been extended so that this works on angles that are not smooth

2.6 Frenet Frame Issues

Sometimes κ is 0. This results in a straight line. The derivative thus becomes 0. This makes the Frenet Frame tricky to calculate, because we can't divide by zero.

2.7 Old-School Curves and Approximations

We have an image of a Bezier curve here.

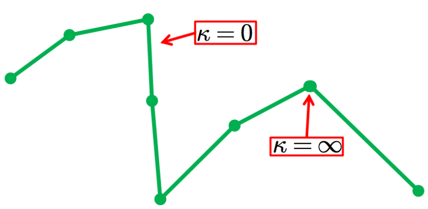


This approach is problematic. Calculus does not apply at the joints of the old-school approach. We have problems dealing with continuities. Getting the closed-form expression of arc length is usually impossible. When they do exists, it is usually messy.
We can only approximate curves.

$$(\text{Bezier curves}) \subseteq (\gamma : R \to R^3) \tag{9}$$

Given a curve, we can approximate it using an arbitrarily close Bezier curve.

We can use a piecewise linear approximation below



We approximate it using line segments
Approximate using a piecewise linear strategy, if we get good enough points, we can get a good approximation. However, the curvature of the piecewise linear curve is 0 at flat points, and infinite at sharp points, which gives us problems differentiating. Thus, we can use

$$f'(x) \approx \frac{1}{h}(f(x+h) - f(x)) \tag{10}$$

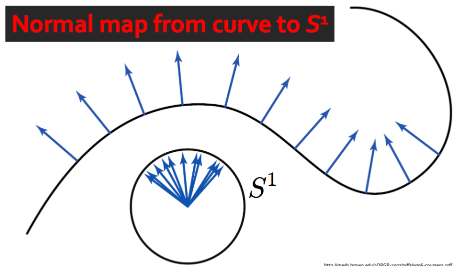
Theorem: As $\Delta h \to 0$, something happens. However, it is hard to simulate $h = 0$ on a computer. If we have a really big piecewise linear approximation curve, it might not converge This, we have to consider two things

1. We can want to make sure our approximations converge.
2. We want to know what happens when h is positive, and discretize them.

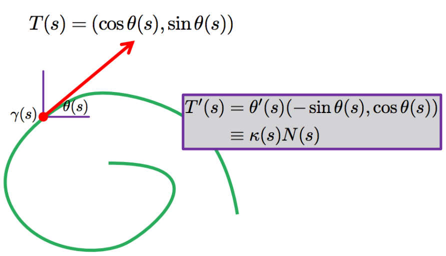
3 Discrete Theories

There is no best theory. It all depends on the application.

3.1 Gauss Map



This is a good model for planar curves.
To get the unit normal of the curve, we just get the tangent and rotate it 90 degrees, and there are only two directions we could do this in.
Gauss Map takes the normals of the curves and maps it to a point on the unit circle. Function from curves to circles. Unit circle can be represented by S^1 because it is a one-dimensional object.

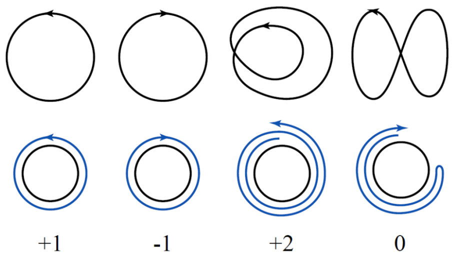


We put an x, y axis for reference, and say that θ is the angle between the x-axis and the line. We can write the tangent as

$$\begin{aligned} T(s) &= (cos\theta(s), \sin\theta(s)) \\ T'(s) &= \theta'(s)(-\sin\theta(s), \cos\theta(s)) \\ &= \kappa(s)N(s) \end{aligned}$$

This number can be negative. This is called the signed curvature.

3.2 Turning Number



The turning number means how many times the curve has looped around itself before you get back to where you started.
This is how we retrieve the turning number.

$$theta'(s) = curvature(s)\Delta\theta = \int_{s0}^{s1} \kappa(s)ds$$

To calculate the change in angle, we integrate the curvature.

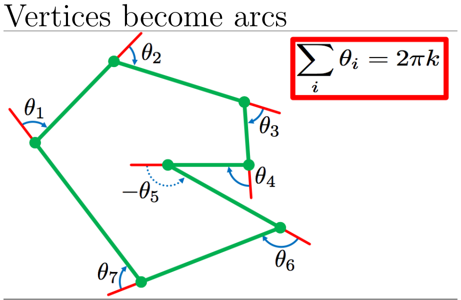
3.3 Turning Number Theorem

$$\int_{\Omega} \kappa(s)ds = 2\pi k \quad (11)$$

2π is 360 degrees. If we came back where we started, we would have had to do a full loop.

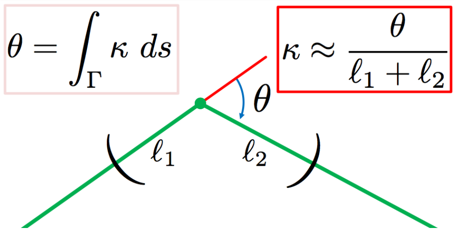
3.4 Discrete Gauss Map

Edges become points
Vertices become arcs



The Gauss Map is constant for a really long time, then has a quick arc, then is constant for a really long time.
We know from high school math that the sum of all the angles should be

$$\sum_i \theta_i = 2\pi k \quad (12)$$

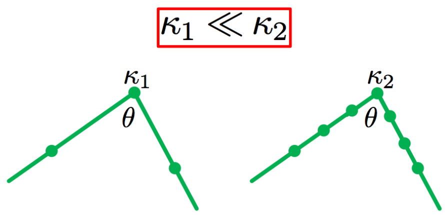


$$\theta = \int_\Gamma \kappa ds \tag{13}$$

We can approximate curvature also by

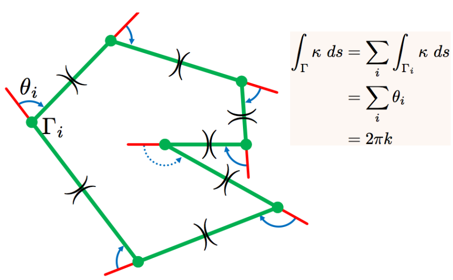
$$\frac{\theta}{\ell_1 + \ell_2} \tag{14}$$

θ is the total change in curvature.
However, our approximation curvature can heavily depend on our step size.



We see this problem occur in the image.

3.5 Discrete Turning Angle Theorem



$$\begin{aligned} \int_\Gamma \kappa(s) ds &= 2\pi k \\ &= \sum_i \int_{\Gamma_i} \kappa ds \\ &= \sum_i \theta_i \\ &= 2\pi k \end{aligned}$$

The idea is that we add the sum of the integration of each segment, which we know how to do.
In this case, curvature is the change of the angles. This holds at every single granularity.
$-\kappa N$ decreases the length the fastest.

The norm is $\nabla L = 2\sin\left(\frac{\theta}{2}\right)$. The norm does not preserve the turning angle, but it preserves a different property, it decreases the length the most when you go in this direction.

We now have a problem, which is in the discrete case, do we want to preserve the turning angle theorem of the property that the norm preserves?
For small enough θ, $2\sin\left(\frac{\theta}{2}\right) = 2 \cdot \frac{\theta}{2} = \theta$

The discrete curvature converges in limit. However, this opens a lot of questions like what type of convergence, and the class of curves.

BIG TAKEAWAY FOR DISCRETE DIFFERENTIAL GEOMETRY
Different discrete behavior like $2\sin\left(\frac{\theta}{2}\right)$ and $2\frac{\theta}{2}$, but same convergence to the differential quantity that we care about.

4 Discrete Elastic Rods

An example of an elastic rod is like hair, shoelace. They have interesting properties when you hold one end fixed and twist from the other. However, the piecewise linear curve does not encode any information about twist.
To simulate objects that are elastic like rope and string, we use another construct.

4.1 Adapted Frame Curve

The material frame is

$$t = \gamma', m_1, m_2 \tag{15}$$

The m_i are our normals to the curve. That way, as long as we know t, we can keep track of these vectors.

4.2 Bending Energy

Rods have potential bending energy. This punishes curvature.

$$E_{bend}(\Gamma) = \frac{1}{2}\int_{\Gamma} \alpha\kappa^2 ds \tag{16}$$

We have to construct our basis in the new frame

$$Kn = t'$$
$$= (t' \cdot t)t + (t' \cdot m_1)m_1 + (t' \cdot m2)m_2$$

We have to understand that t, m_1, m_2 form an orthonormal basis for 3-space.
Every addend of the form $(t' \cdot t)tor(t' \cdot m_1)m_1$ is a projection on the vector on the right-hand side.

$$= (t'\dot{m}_1)m_1 + (t'\dot{m}_2)m_2$$
$$= \omega_1 m_1 + \omega_2 m_2$$

Important note T is zero in the equation above.

4.3 Twisting Energy

Rods also have potential twisting energy

$$E_{twist}(\Gamma) = \frac{1}{2}\int_{\Gamma} \beta m^2 ds \tag{17}$$

This punishes non-tangent change in frame

$$m = m_1' \cdot m_2$$

We know that $m_1 \cdot m_2$ is zero because m_1 and m_2 are orthonormal.

$$= \frac{d}{dt}(m_1 \cdot m_2) - m_1 \cdot m_2'$$
$$= -m_1 * m_2'$$

Derivative of m_1 has two components. One is parallel to m_2 and the other is parallel to the tangent. The component changing that is parallel to tangent is basically the curvature of curve. So we really want to measure the twist which is orthogonal to the tangent.

4.4 Which Basis To Use

DRAWBACKS OF FRENET FRAME
-not defined for straight lines
-intuition of binormal versus normal is fuzzy

We have an alternate frame called the Bishop frame

5 Bishop Frame

-most relaxed frame for the surface
-solve a simple differential equation in order to use
What do we mean by relaxed?
0.draw straight line down the shoelace 1.take your shoelace and feed it through the straw 2.bend your straw into the shape you care about 3.hold your shoelace fixed 4.let rest of the shoelace relax The unit tangent points down the tube of the straw and we get now the line we drew is twisting around the curve. We get two other orthogonal vectors twisting around the straw. This is the Bishop frame. We can get an ODE to compute the Bishop frame. The derivatives of the vectors are parallel to the cross product of the original frame and the Darboux vector.

$$t' = \Omega \cdot t \tag{18}$$
$$u' = \Omega \cdot u \tag{19}$$
$$v' = \Omega \cdot v \tag{20}$$

$$\Omega = \kappa B \tag{21}$$

5.1 Curve-Angle Representation

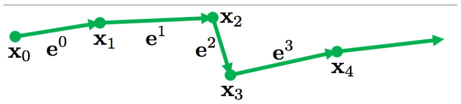
We have a curve and we're going to compute u, v at every segment of the curve. Then, we'll have an angle θ that will rotate us to a twisted version of the curve. Then we just how much we've twisted in relation to the most relaxed version of the frame.

$$m_1 = u\cos\theta + v\sin\theta$$
$$m_2 = -u\sin\theta + v\cos\theta$$

$$E_{twist}(\Gamma) = \frac{1}{2}\int_\Gamma \beta(\theta')^2 ds \tag{22}$$

Choice of Bishop frame is awesome because all we need now is θ twisting energy

5.2 Discrete Kirchoff Rods



$$\mathbf{t}^i = \frac{\mathbf{e}^i}{\|\mathbf{e}^i\|}$$

x's are joints of the curve.
Compute the tangents and tangents live on the edges.

$$t_i = \frac{e^i}{\|e^i\|} \tag{23}$$

Now, we have another curvature

$$\kappa_i = 2\tan\frac{\phi_i}{2} \tag{24}$$

As ϕ gets small, tan converges to the same thing as the previous ones.

$$(\kappa b)_i = \frac{2e^{i-1} \times e^i}{\|e^{i-1}\|\|e^i\| + e^{i-1}\cdot e^i} \tag{25}$$

The binormal cross product with the tangent is the norm of the adjacent edges.
Bending Energy:
We cheat a little big because curvature lives on edge, but we need it on a vertices. The curvature binormal lives on the vertices.

$$E_{bend}(\Gamma) = \frac{\alpha}{2}\sum_i \left(\frac{(\kappa b)_i}{\ell_i/2}\right)^2 \frac{\ell_i}{2}$$
$$= \alpha \sum_i \frac{\|\kappa b)_i\|^2}{\ell_i}$$

Our Bending energy is just the integral of the curvature, so the above equation tries to turn it into a point wise quantity, by dividing it by the length of the segment, then we square it to get a good approximation.
The norm of $\kappa B = \kappa$. The above equation is an approximation the bending energy.
This all just an approximation.

5.3 Discrete Parallel Transport

How do we define the bishop frame?
We just u and v, and use the parallel transport to bring to the next point.
Take vector and drag it along a curve. Example, the normal vector must be perpendicular to the curve at all times. The parallel transport does it in the most relaxed way possible. It helps us bring two orthogonal vectors on one tangent to their matching version on another tangent.

$$P_i(t^{i-1}) = t^i \tag{26}$$

$$P_i(t^{i-1} \times t^i) = t^{i-1} \times t^i$$

$$\begin{aligned} u^i &= P_i(u^{i-1}) \\ v^i &= t^i \times u^i \end{aligned}$$

5.4 Discrete Material Frame

$$\begin{aligned} m_1^i &= u^i \cos\theta^i + v^i \sin\theta^i \\ m_2^i &= -u^i \sin\theta^i + v^i \cos\theta^i \end{aligned}$$

5.5 Discrete Twisting Energy

$$E_{twist}(\Gamma) = \beta \sum_i \frac{(\theta^i - \theta^{i-1})^2}{\ell_i} \tag{27}$$

θ can be arbitrary because we only care about the derivative.
We have three orthonormal vectors tangent, u and v, and the θ tells us how much we are twisting from the most relaxed frame.

5.6 Real Life Physics

If you pour honey on treadmill, it behaves like a string. Thus, our simulation for rods can also work for fluids in certain cases.

6 Morals

For one curve, we can get three curvatures

$$\begin{pmatrix} \theta 2 & \sin\frac{\theta}{2} & 2\tan\frac{\theta}{2} \end{pmatrix} \tag{28}$$

Easy theoretical objects can still be hard to use. Like for example, the Frenet frame is easy theoretically to understand but almost impossible to use.
Proper coordinates and DOFs go a long way. Take the Bishop Frame, which just adds one DOF in θ and allows us to compute bending and twisting. Trying to do this in xyz coordinates would be incredibly hard.