CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 2: Curves

The arc-length re-parametrization

- The question we will answer here is: given any smooth, regular curve $\gamma : I \to \mathbb{R}$, is it possible to find a re-parametrization of γ by arc-length?
- In other words: is it possible to find a smooth bijection $\phi : [0, \mathit{length}(\gamma)] \to I$ so that $\tilde{\gamma} : [0, \mathit{Length}(\gamma)] \to \mathbb{R}$ defined by $\tilde{\gamma}(s) := \gamma \circ \phi(s)$ has constant speed?
- The answer is YES. To see why, define the function $\ell : I \to [0, \mathit{length}(\gamma(I))]$ by

$$\ell(t) := \mathit{length}\big(\gamma([0,t])\big) = \int_0^t \|\dot{\gamma}(x)\| dx$$

- Note that $\frac{d\ell(t)}{dt} = \|\dot{\gamma}(t)\|$. Since γ is regular, there are no points where $\dot{\gamma} = 0$. Hence $\frac{d\ell(t)}{dt}$ never vanishes, so that ℓ is invertible.
- Define the re-parametrization $\phi = \ell^{-1}$. So in other words, we introduce the new parameter s satisfying $s = \ell(t)$ or $t = \ell^{-1}(s) =: \phi(s)$. We let $\tilde{\gamma}(s) := \gamma(\phi(s))$.
- Now we can show that $\|\frac{d}{ds}\tilde{\gamma}(s)\| = 1$ as follows:

$$\frac{d\tilde{\gamma}(s)}{ds} = \frac{d\gamma}{dt} \circ \phi(s) \frac{d\phi(s)}{ds} = \dot{\gamma} \circ \phi(s) \frac{d\ell^{-1}(s)}{ds} = \frac{\dot{\gamma} \circ \ell^{-1}(s)}{\frac{d\ell}{dt} \circ \ell^{-1}(s)} = \frac{\dot{\gamma} \circ \ell^{-1}(s)}{\|\dot{\gamma} \circ \ell^{-1}(s)\|}$$

- Thus $\|\frac{d\tilde{\gamma}(s)}{ds}\| = 1$ and the re-parametrized version is parametrized by arc-length.

Arc-length example calculation

- Sadly, the arc-length function ℓ for most curves does not lead to a nice closed form expression. (Try it! Even something simple like $\gamma(t) := (t, t^2)$ is already problematic.) And then even if we manage to find ℓ, it may be impossible to find a nice expression for ℓ^{-1}.
- The arc-length parametrization is very useful theoretically (as we'll see) but difficult to work with in practice.
- A doable example is provided by the logarithmic spiral: $\gamma(t) = (e^t \cos(t), e^t \sin(t))$. We have:

$$\dot{\gamma}(t) = e^t(\cos(t), \sin(t)) + e^t(-\sin(t), \cos(t))$$

  and so

$$\|\dot{\gamma}(t)\| = e^t \|(\cos(t), \sin(t)) + (-\sin(t), \cos(t))\| = \sqrt{2}e^t\,.$$

  Consequently,

$$\mathit{length}(\gamma([0,T])) = \int_0^T \|\dot{\gamma}(t)\| dt = \sqrt{2}\int_0^T e^t dt = \sqrt{2}(e^T - 1\,.)$$

- Hence $s = \ell(t) := \sqrt{2}(e^t - 1)$. Therefore $t = \ell^{-1}(s) := \log(s/\sqrt{2} + 1)$
- Hence the re-parametrized version of the logarithmic spiral is

$$\tilde{\gamma}(s) = \left(\frac{s}{\sqrt{2}} + 1\right)\big(\cos(\log(s/\sqrt{2}+1)), \sin(\log(s/\sqrt{2}+1))\big)\,.$$

- You can check that this curve has constant velocity equal to 1.