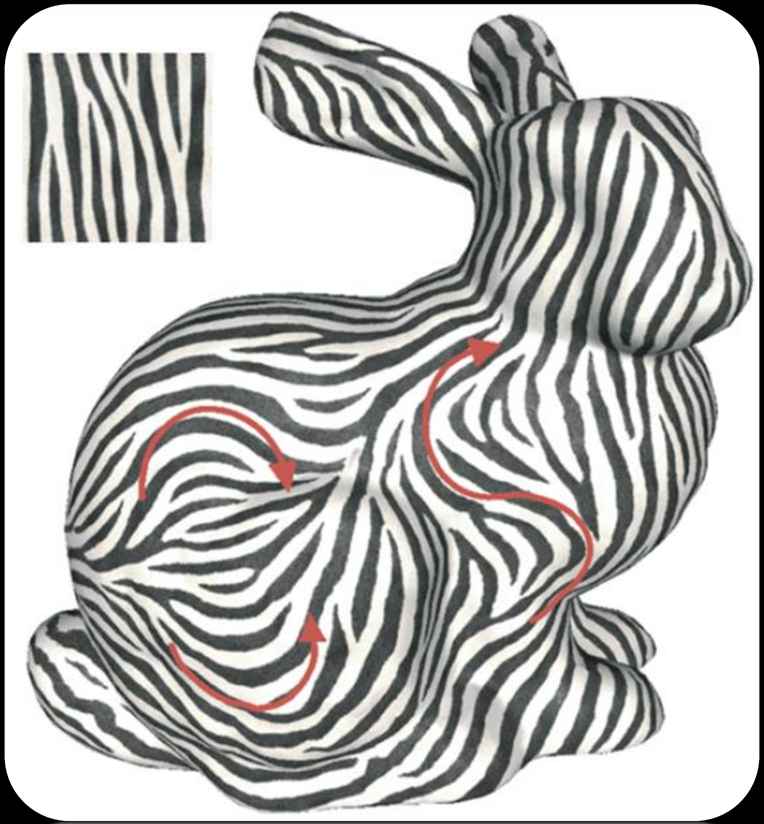
Fisher et al. "Design of Tangent Vector Fields" (SIGGRAPH 2007)



Discrete Exterior Calculus

CS 468, Spring 2013

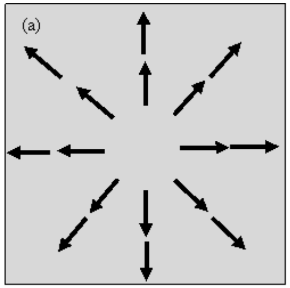
Differential Geometry for Computer Science

Justin Solomon and Adrian Butscher

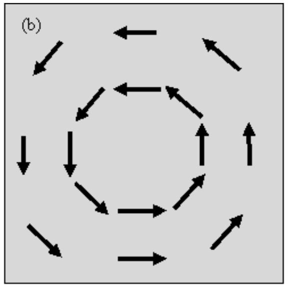
<math_review>

Vector Calculus

$$\text{div } \vec{v} \equiv \nabla \cdot \vec{v} \equiv \sum_i \frac{\partial v_i}{\partial x_i}$$



$$\text{curl } \vec{v} \equiv \nabla \times \vec{v} \equiv \cdots$$



$$\Delta f \equiv \nabla \cdot \nabla f \equiv \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

Famous Theorems (in R^2)

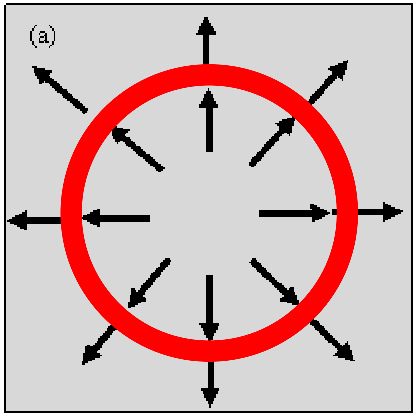
$$\int \text{div}\ \vec{v}\ dA = \int_{\partial} \vec{v} \cdot \vec{n}\ dl$$

"Divergence Theorem"

$$\int \text{curl}\ \vec{v}\ dA = \int_{\partial} \vec{v} \cdot \vec{t}\ dl$$

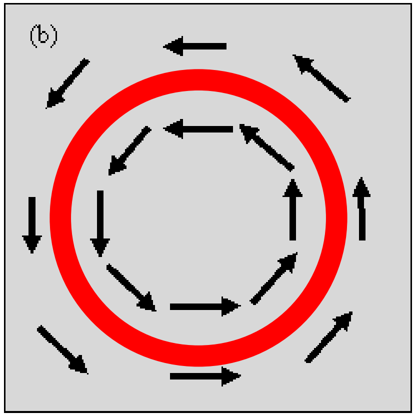
"Green's Theorem"

Famous Theorems (in R^2)



$$\int \text{div}\, \vec{v}\, dA = \int_{\partial} \vec{v} \cdot \vec{n}\, dl$$

"Divergence Theorem"



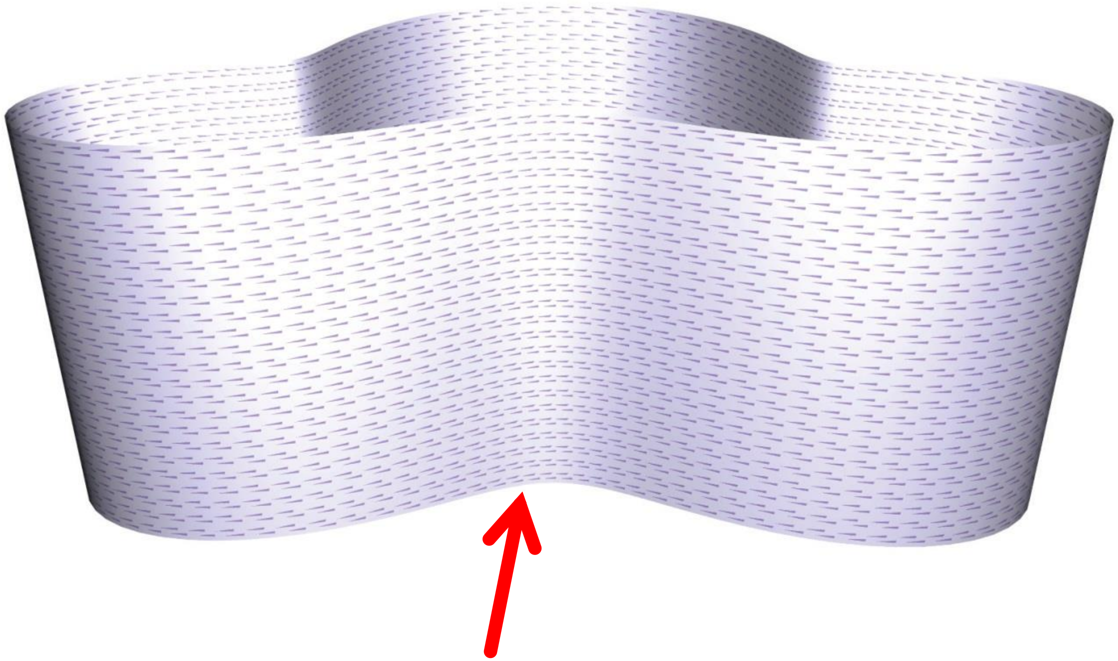
$$\int \text{curl}\, \vec{v}\, dA = \int_{\partial} \vec{v} \cdot \vec{t}\, dl$$

"Green's Theorem"

Exterior Calculus

Extension of vector calculus to surfaces (and manifolds).

New Rule



Everything must be **intrinsic**!

Vector fields are **tangent**!

Differential Forms

For each point *p* on a surface:

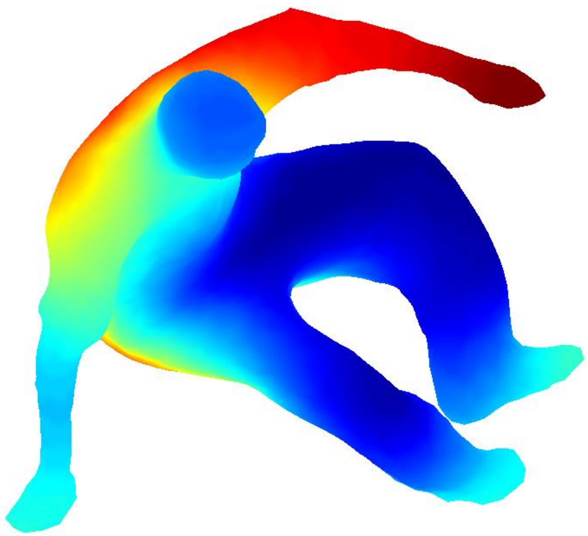
k vectors in the tangent space at *p* → ***Differential k-form*** → $\mathbb{R}$

k-linear

Alternating

[Sanity check: In *n* dimensions, *p*-forms are zero for $p > n$.]

Easiest Example



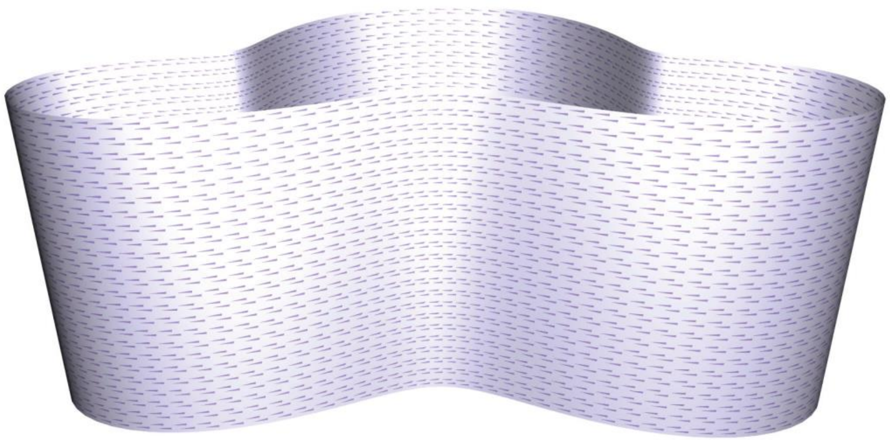
$$f : \Sigma \to \mathbb{R}$$

↓

0-form

http://perception.inrialpes.fr/Publications/2011/ZBH11a/scale-9.png

Differential One-Forms



Vector field

$$\vec{v} : \Sigma \to T\Sigma \subset \mathbb{R}^3$$

$$\vec{v}^\flat \downarrow \uparrow \omega^\sharp$$

1-form ω:

$$\omega(\vec{x}) = \vec{v} \cdot \vec{x}$$

Trivia: Musical Isomorphisms

$$\omega^i = \sum_j g^{ij} v_j$$

Elgar, Cello Concerto

Sharp operator *raises* indices

Trivia: Musical Isomorphisms

$$v_i = \sum_j g_{ij}\omega^j$$

Bloch, Schelomo

Flat operator *lowers* indices

Evaluating One-Forms

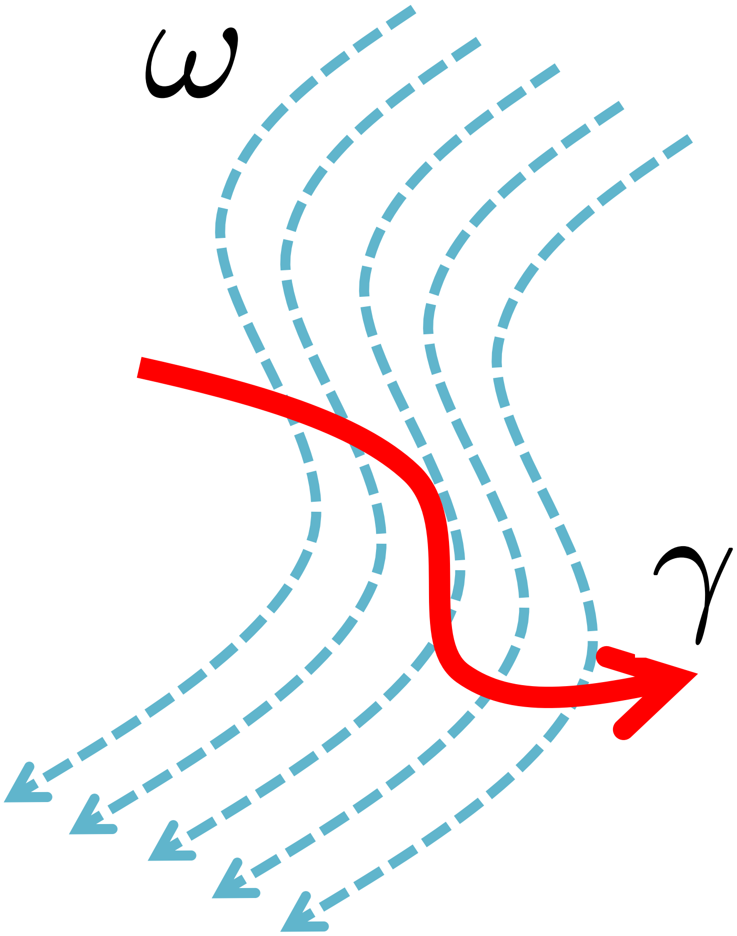
$$\omega(\vec{v}) = \sum_i \omega^i v_i$$

No metric matrix *g*

Zoo of Operators

$\omega^\sharp$	1-form to vector
$\vec{v}^\flat$	Vector to 1-form
$d\omega$	Exterior derivative
$\star\omega$	Dual *k*-forms → (*n*-*k*)-form (plane to its normal)
$\omega_1 \wedge \omega_2$	Product of forms *k*,*p*-forms → (*k*+*p*)-form (cross product!)

Integration of *k*-Forms



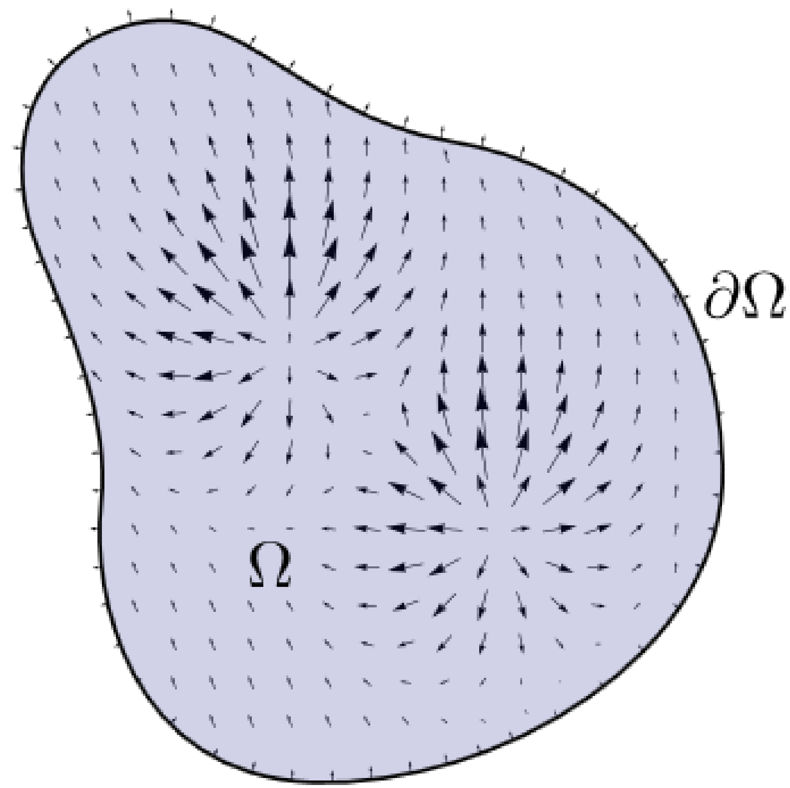
$$\int_\gamma \omega \equiv \int_\gamma \omega(T)\, ds$$

Measures amount of ω parallel to γ

Integrate on *k*-dimensional objects

Stokes' Theorem

$$\int d\omega = \int_{\partial} \omega$$



http://brickisland.net/cs177/wp-content/uploads/2011/11/ddg_divergence_theorem.svg

</math_review>

Discrete Exterior Calculus (DEC)

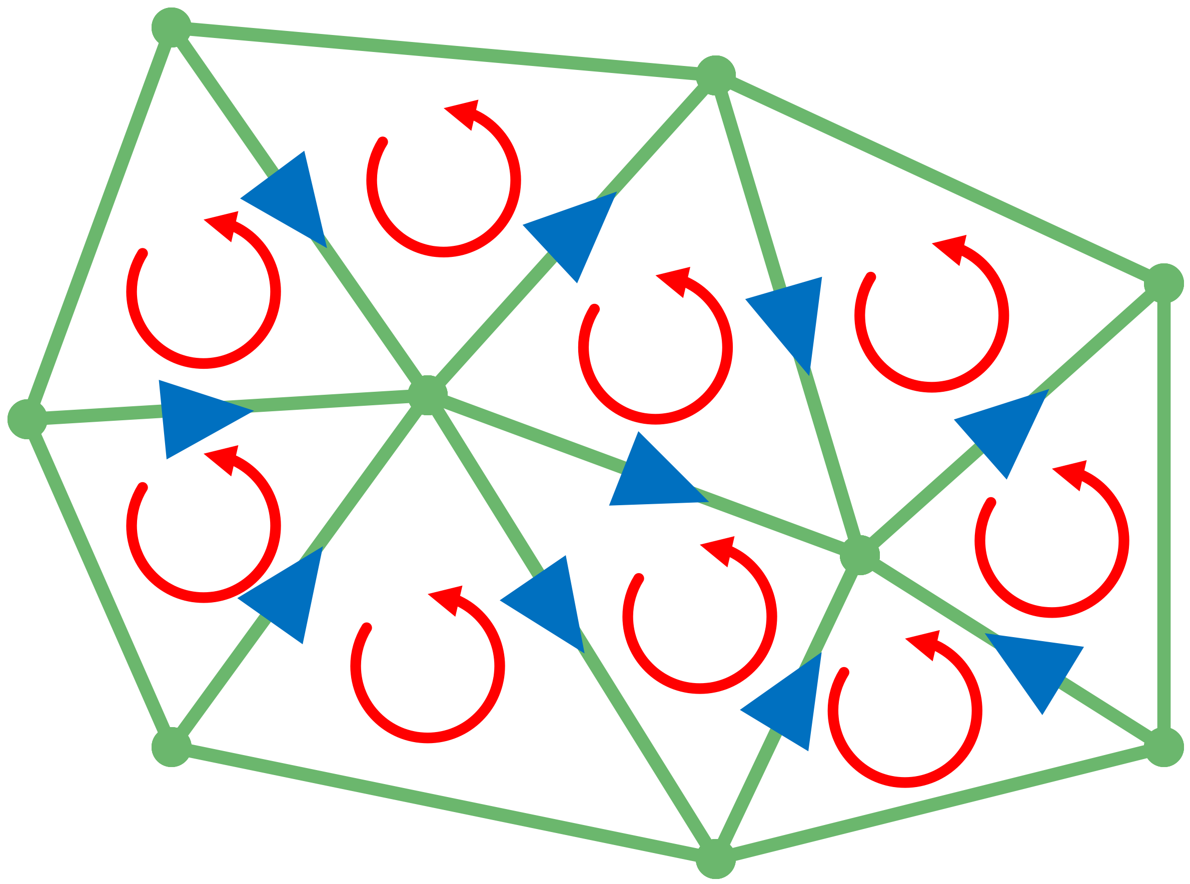
Discrete version of exterior calculus.

$$\omega^\sharp \quad \vec{v}^\flat \quad \omega_1 \wedge \omega_2 \quad \star\omega \quad d\omega$$

...

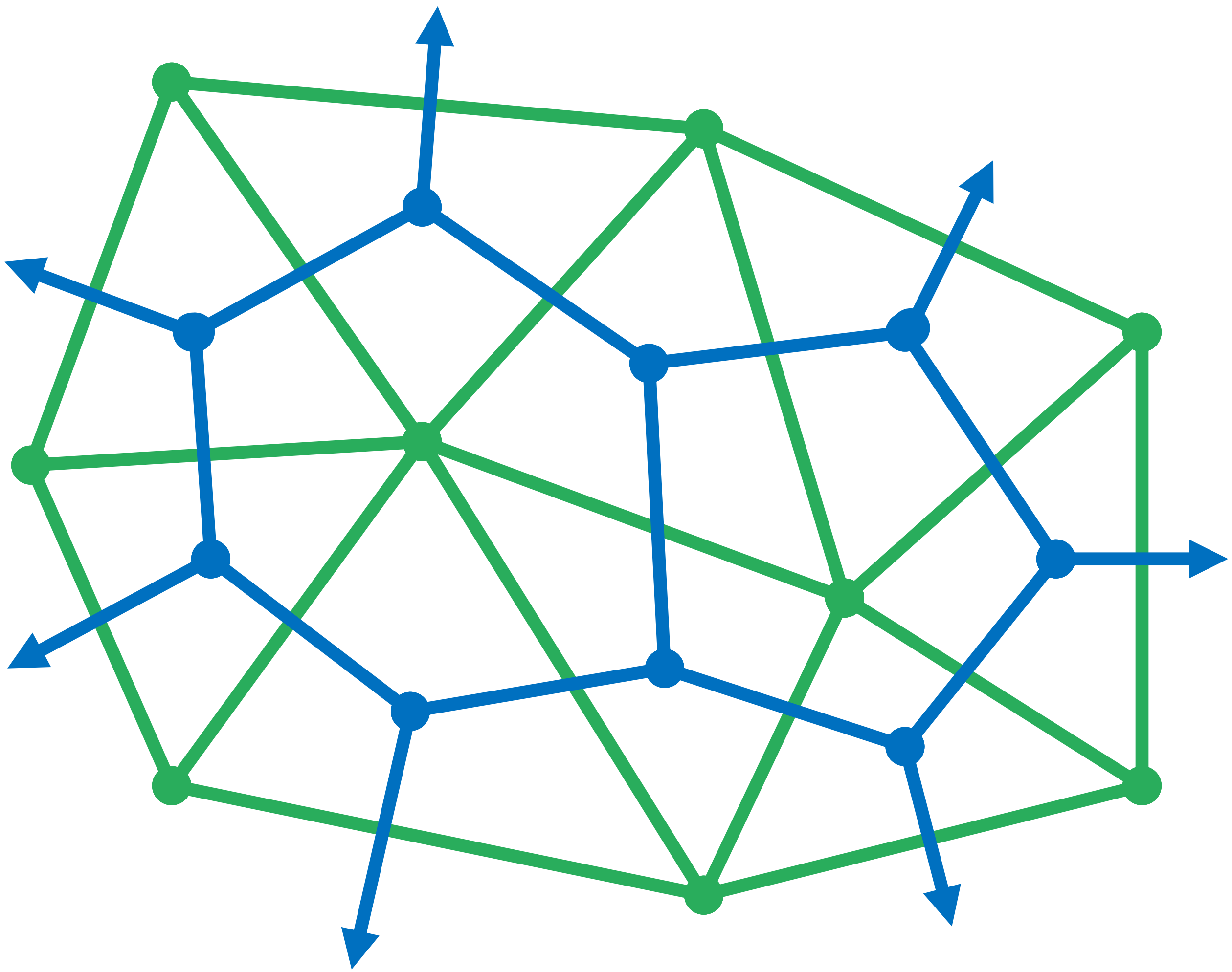
Recall:

Oriented Simplicial Complex



Recall:

Dual Complex

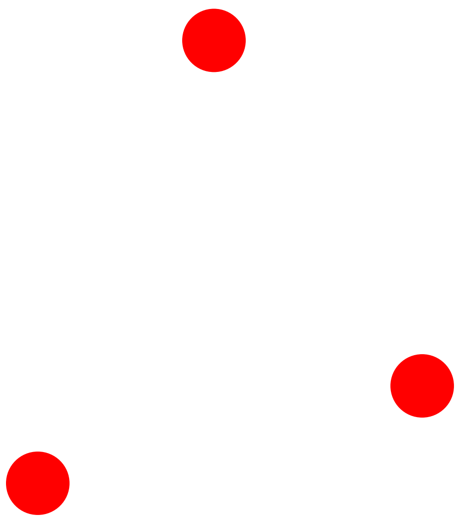


The Trick

Store *integrals* of forms!

The Trick

Discrete 0-form

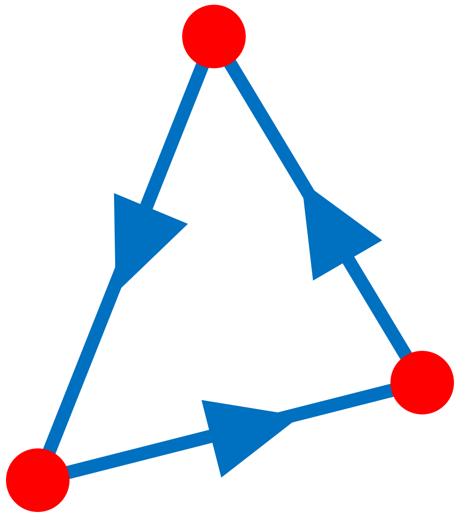


$$\int_v \omega = f(v) \to \mathbb{R}^{|V|}$$

Store *integrated* quantities!

The Trick

Discrete 1-form

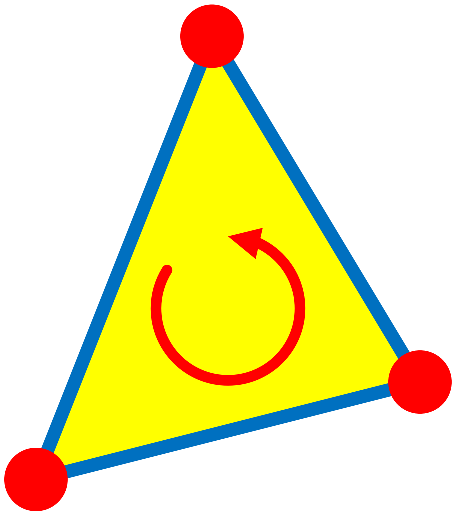


$$\int_e \omega \to \mathbb{R}^{|E|}$$

Store *integrated* quantities!

The Trick

Discrete 2-form



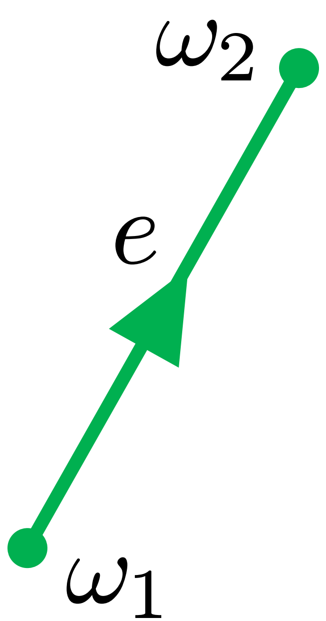
$$\int_t \omega \to \mathbb{R}^{|F|}$$

Store *integrated* quantities!

Exterior Derivative

$$\int d\omega = \int_{\partial} \omega$$

Stokes' Theorem



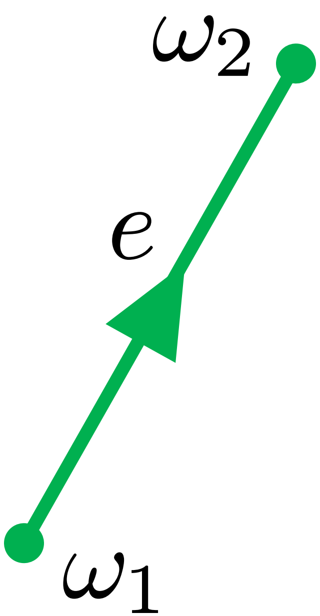
Implemented on homework 3!

$$\int_e d\omega = \int_{\partial e} \omega = \omega_2 - \omega_1$$

Exterior Derivative

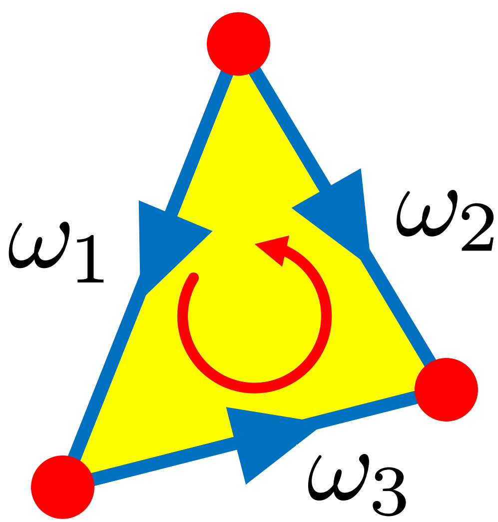
$$d \in \mathbb{R}^{|E| \times |V|}$$

consists of 1, 0, -1



$$\int_e d\omega = \int_{\partial e} \omega = \omega_2 - \omega_1$$

Exterior Derivative

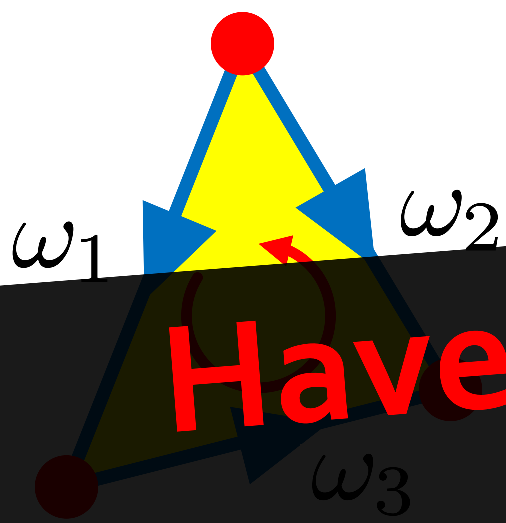


$$d \in \mathbb{R}^{|F| \times |E|}$$

consists of 1, 0, -1

$$\int_t d\omega = \int_{\partial t} \omega = \omega_1 - \omega_2 + \omega_3$$

Exterior Derivative



$$d \in \mathbb{R}^{|F| \times |E|}$$

consists of 1, 0, -1

Haven't made any approximations yet!

$$\int_t d\omega = \int_{\partial t} \omega = \omega_1 - \omega_2 + \omega_3$$

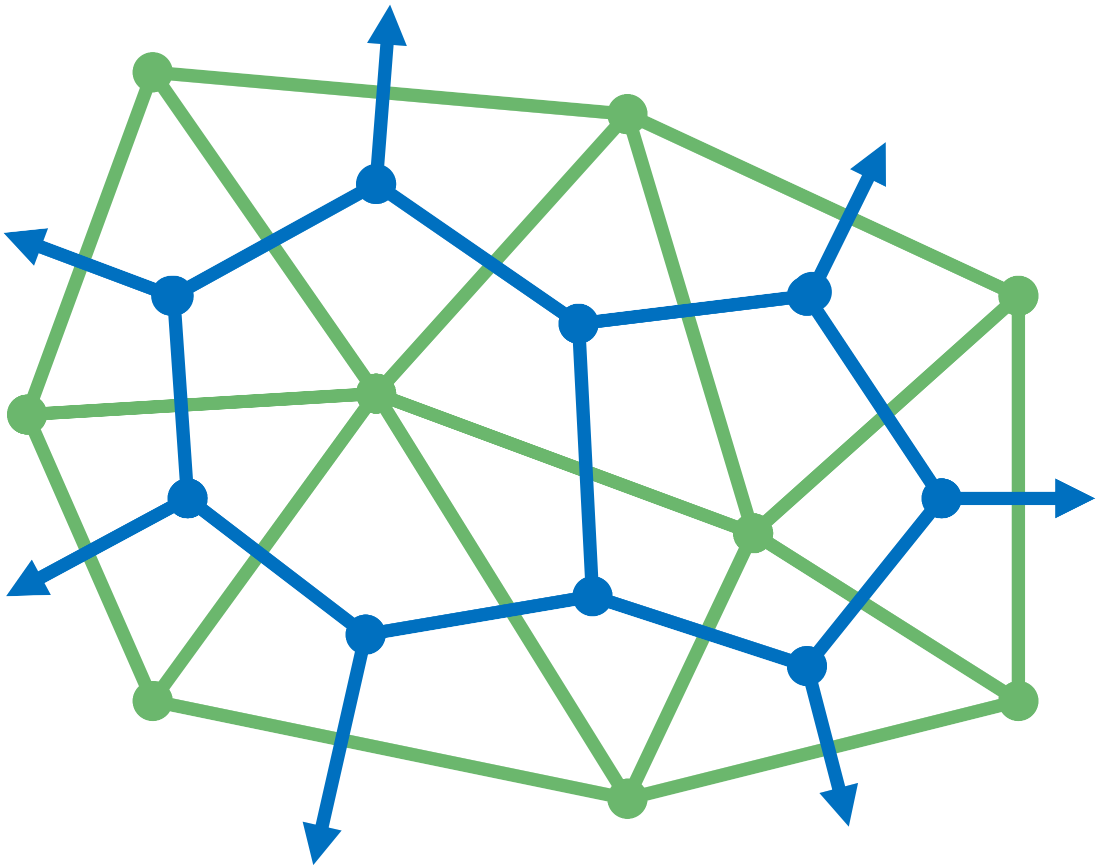
Observation

$$“d^2 = 0”$$

Proved on homework 3!

Two different *d* matrices

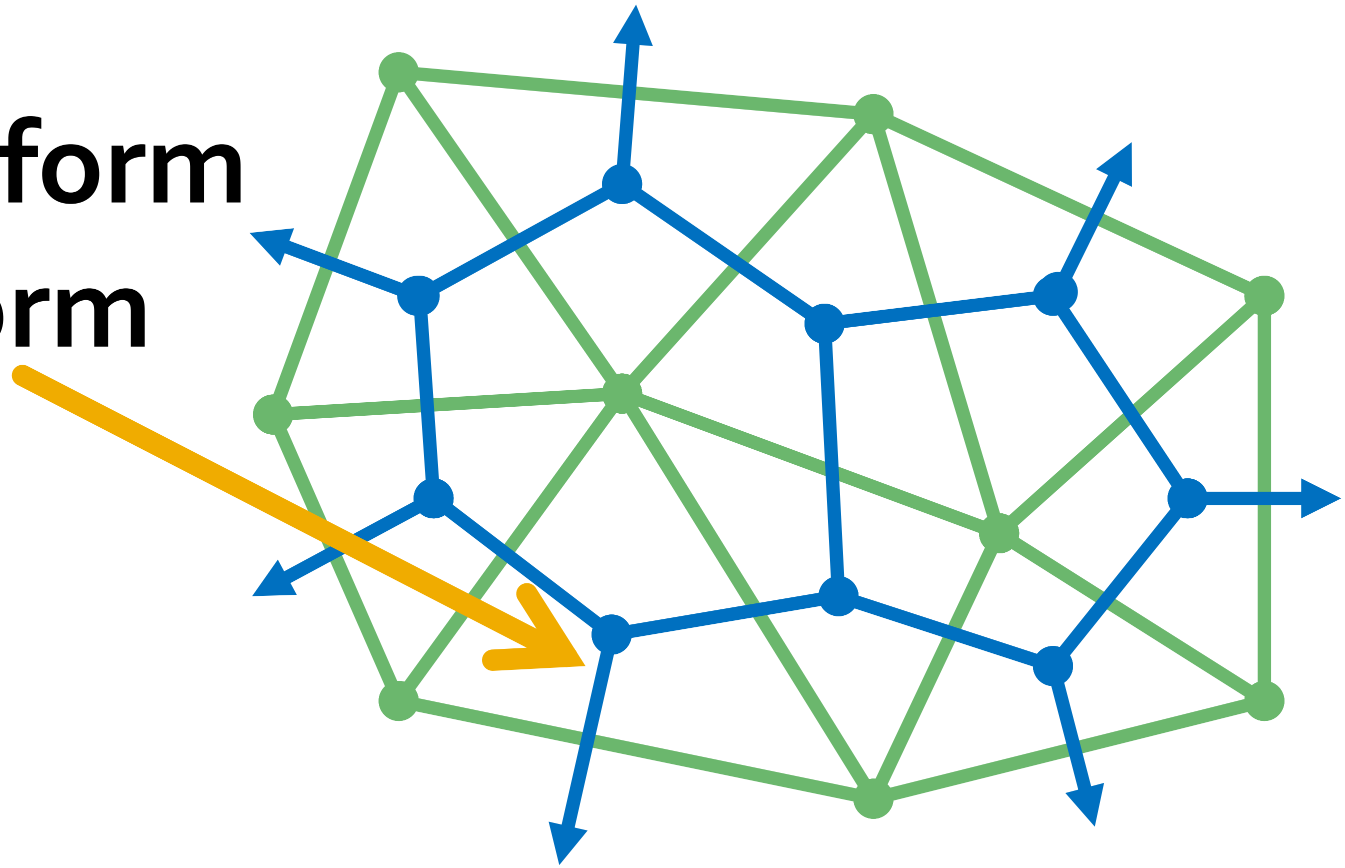
Hodge Star: Idea



Moves to dual mesh

Hodge Star

Primal 2-form
Dual 0-form

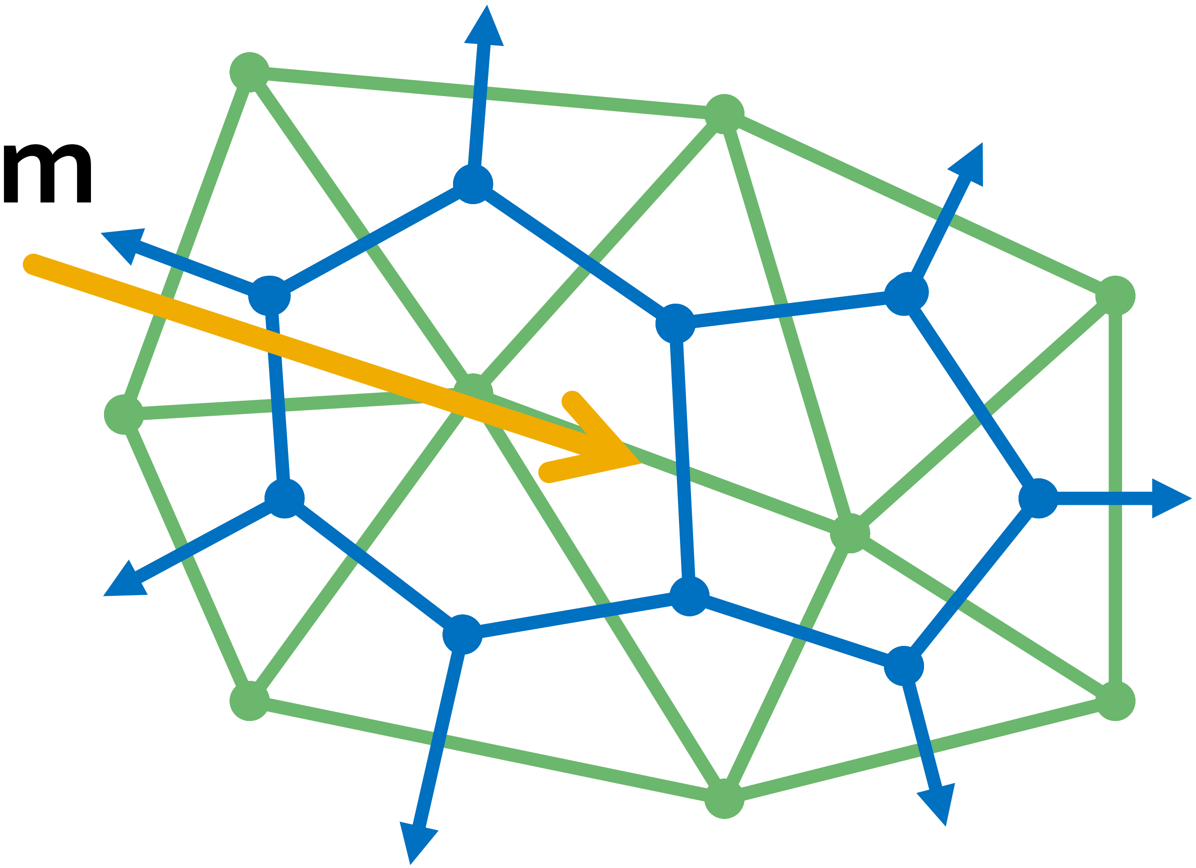


Moves to dual mesh

Hodge Star

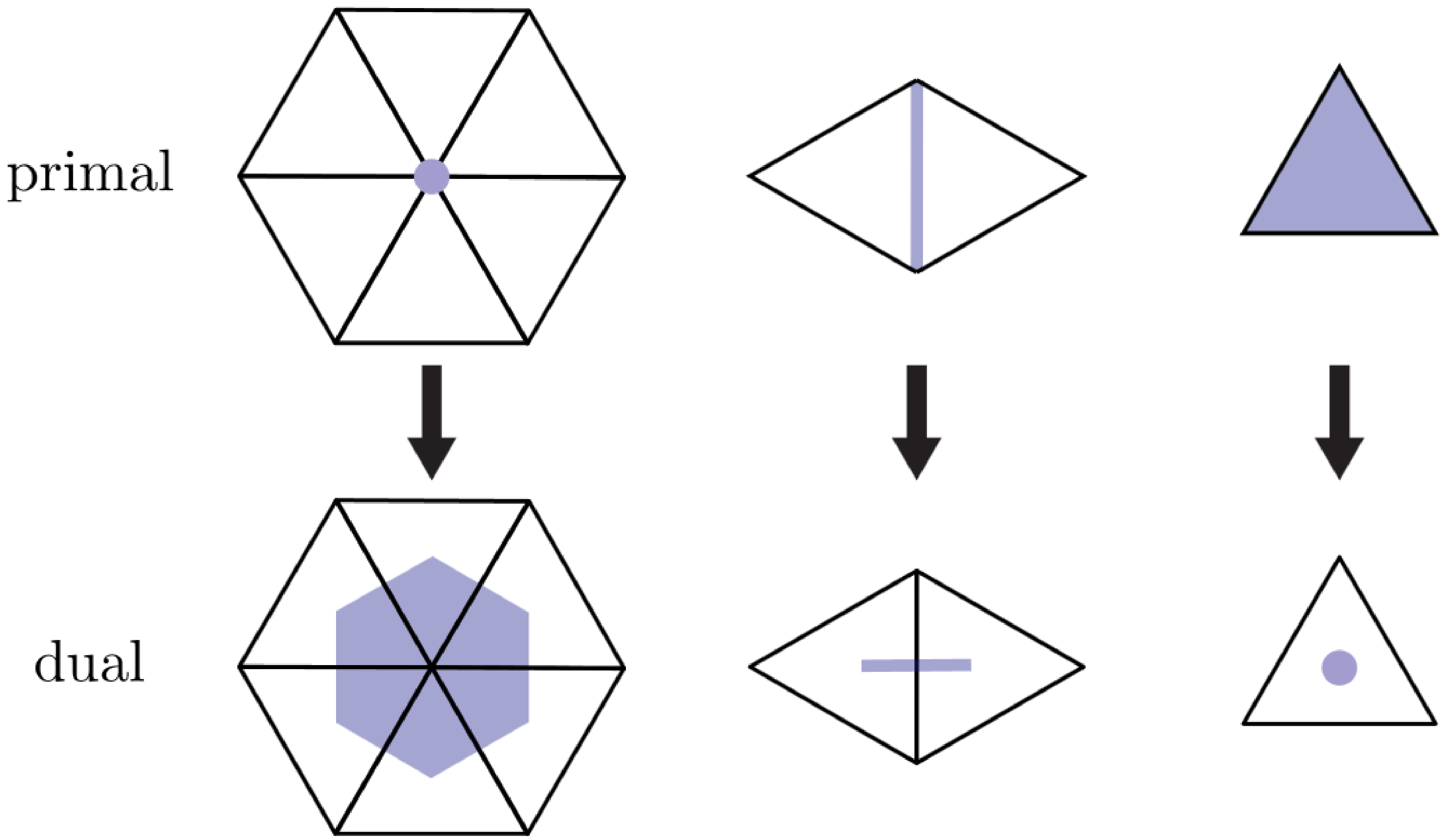
Primal 1-form

Dual 1-form



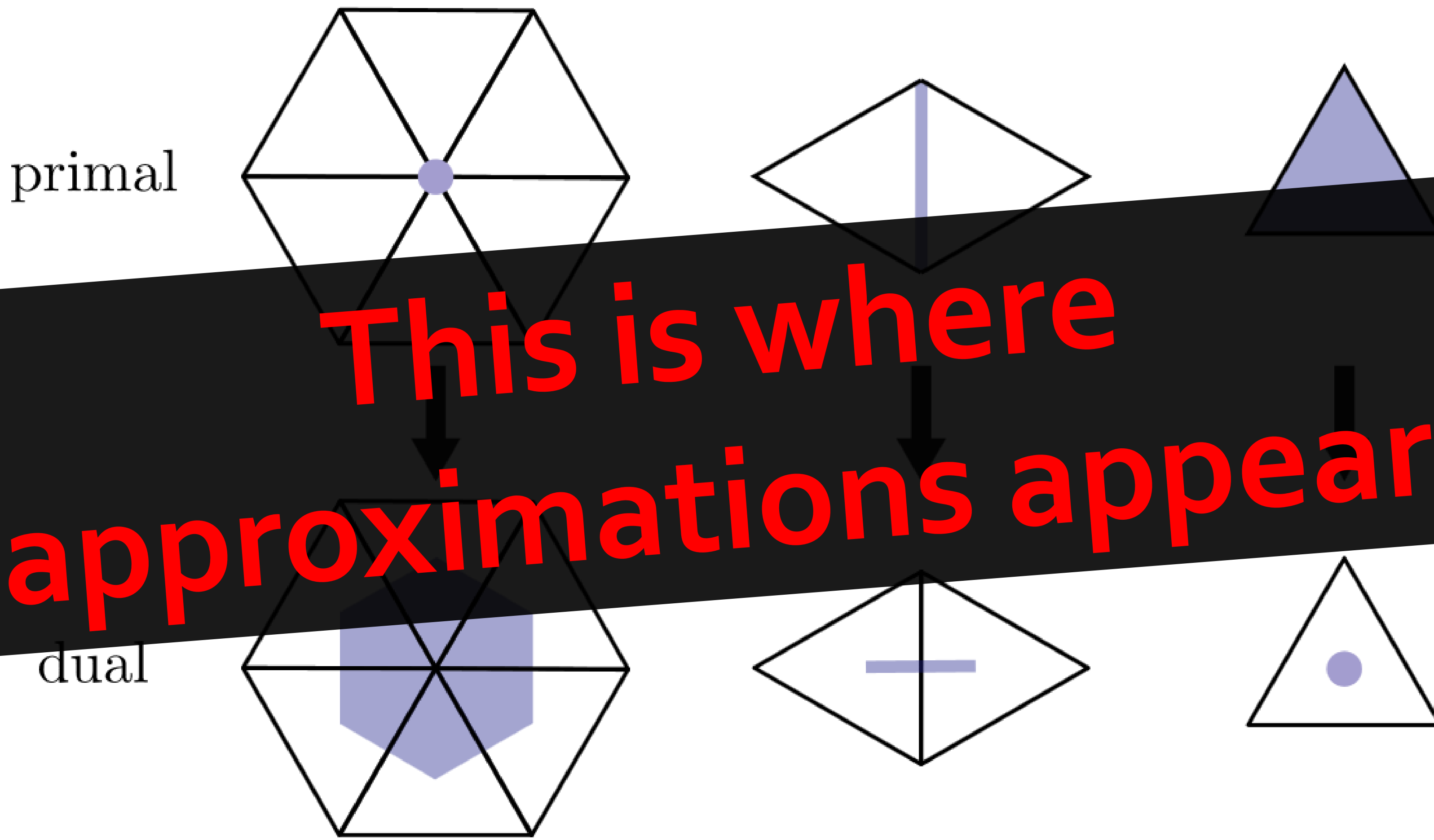
Moves to dual mesh

Hodge Star Matrices



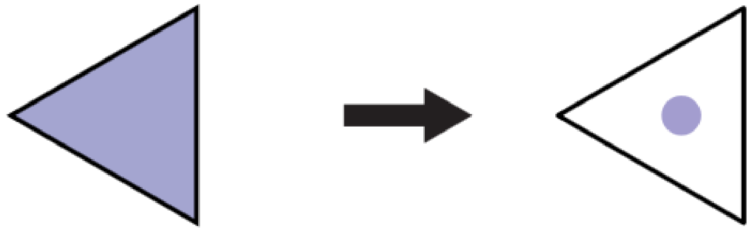
http://brickisland.net/cs177/

Hodge Star Matrices



This is where approximations appear.

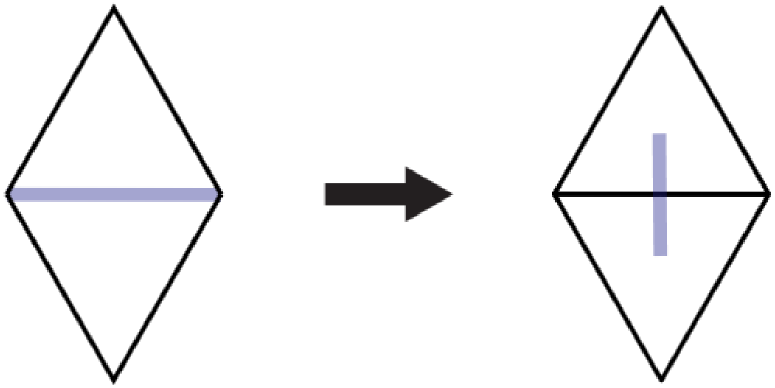
Primal 2-Form / Dual 0-Form

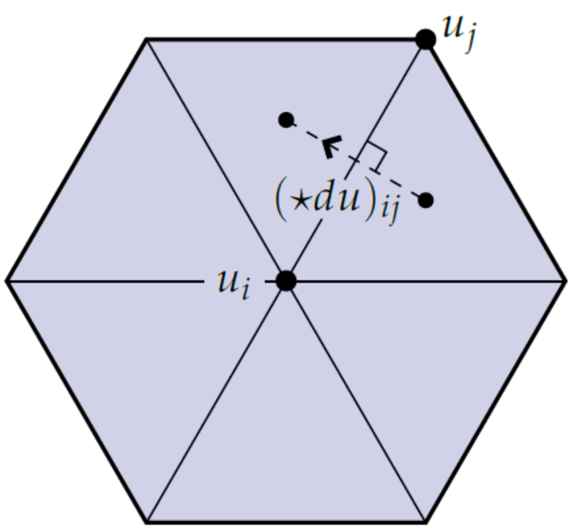


$$\star_{ii} = \mathrm{Area}(\mathrm{triangle}\ i)^{-1}$$

Just triangle areas

Primal 1-Form / Dual 1-Form



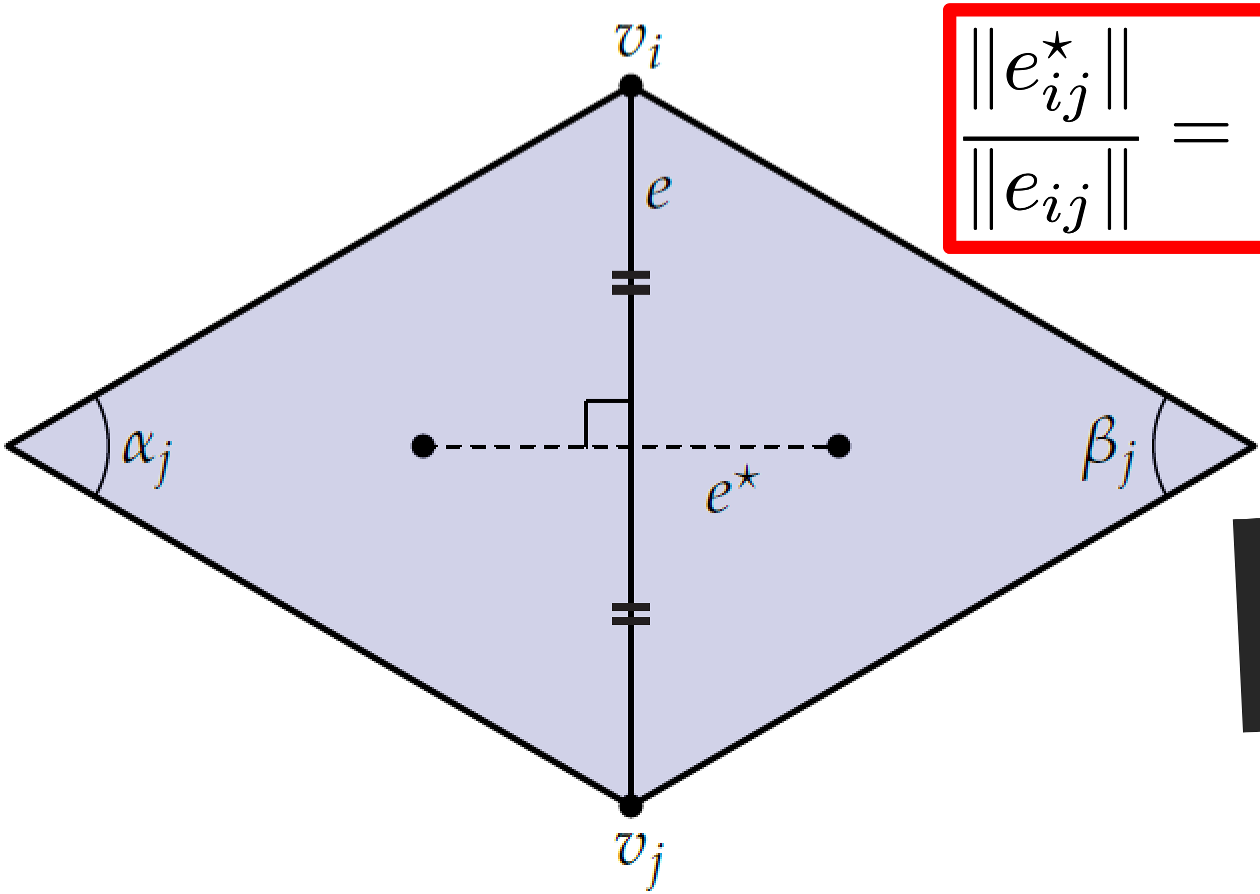


$$\star\omega = \frac{|e_\star|}{|e|}\omega$$

http://users.cms.caltech.edu/~keenan/pdf/DGPDEC.pdf

Ratio of edge lengths

Primal 1-Form / Dual 1-Form



$$\frac{\|e_{ij}^\star\|}{\|e_{ij}\|} = \frac{1}{2}(\cot\alpha_j + \cot\beta_j)$$

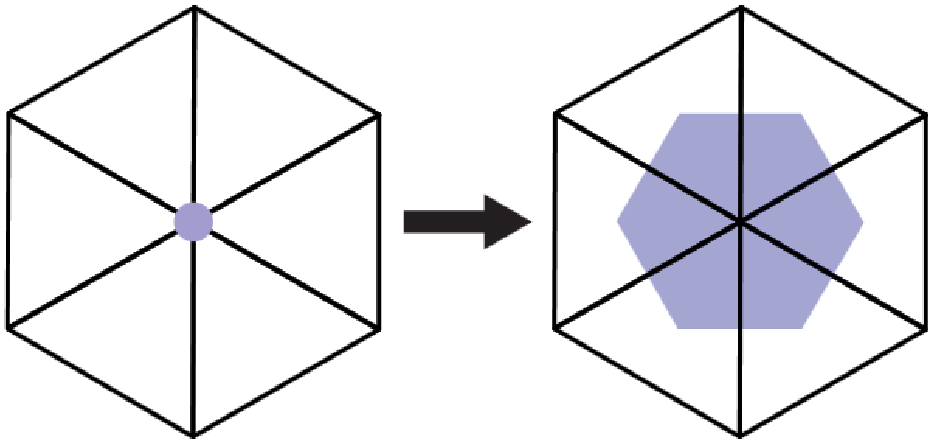
Omit calculation

Enough already!

http://users.cms.caltech.edu/~keenan/pdf/DGPDEC.pdf

Choice of dual: Circumcenter

Primal 0-Form / Dual 2-Form

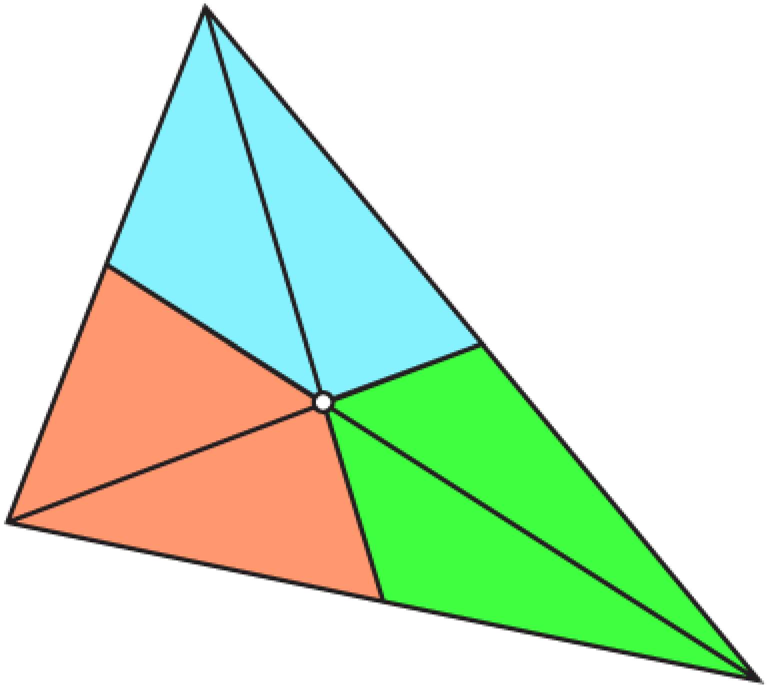


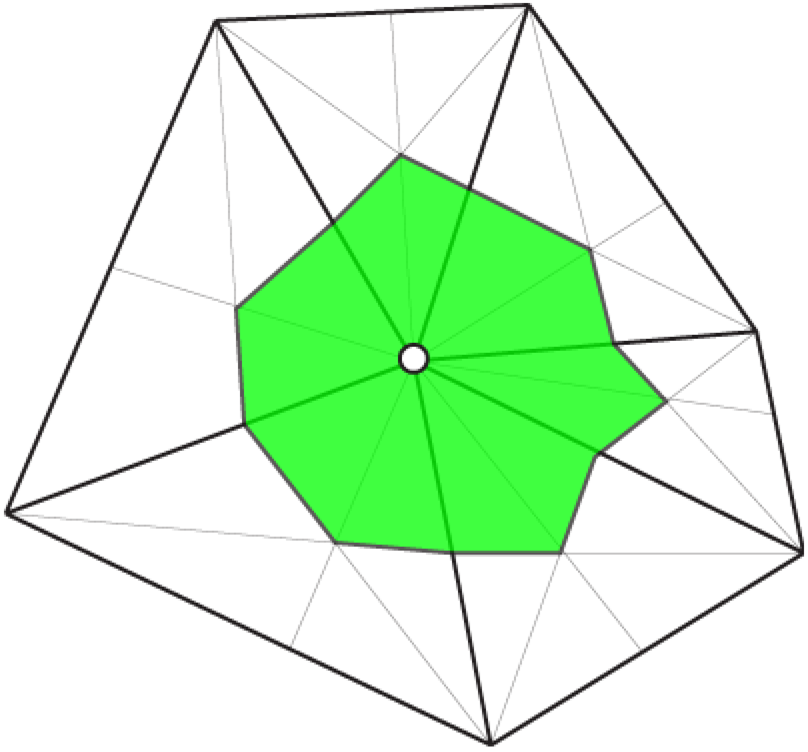
$$\star_{ii} = \text{Area}(\text{cell } i)$$

Area of dual cell

Recall:

Barycentric Lumped Mass

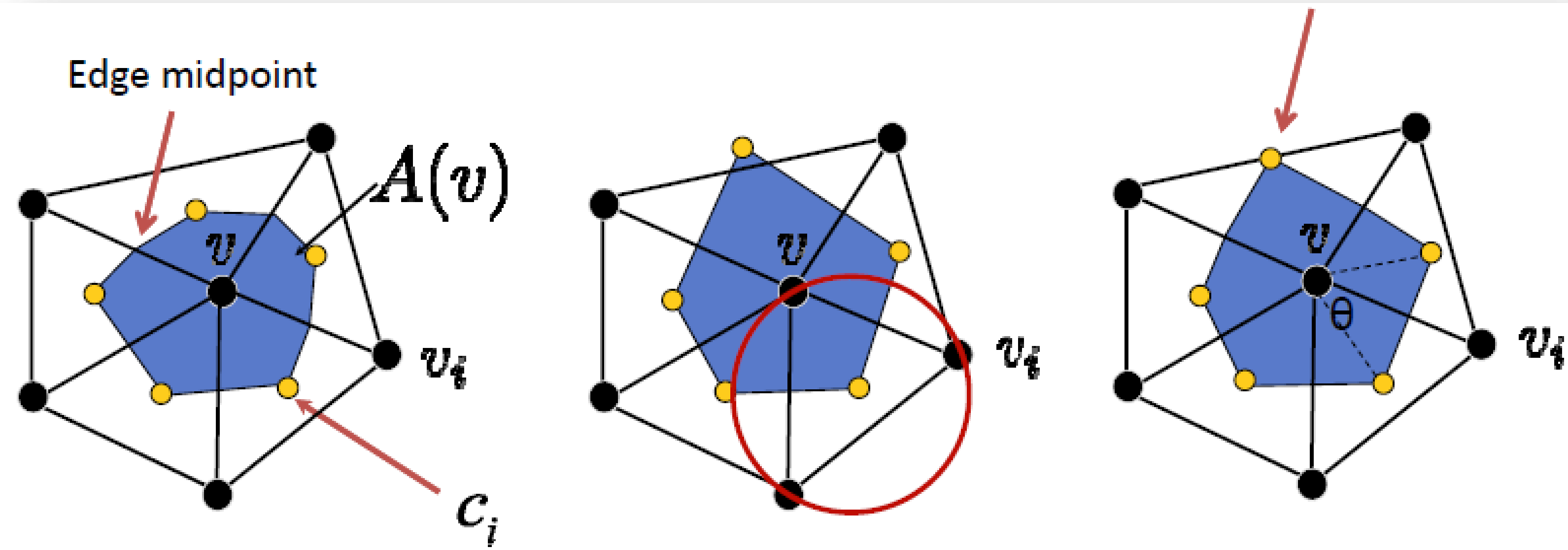




http://www.alecjacobson.com/weblog/?p=1146

Area/3 to each vertex

Additional Options



Barycentric cell

c_i = barycenter of triangle

Voronoi cell

c_i = circumcenter of triangle

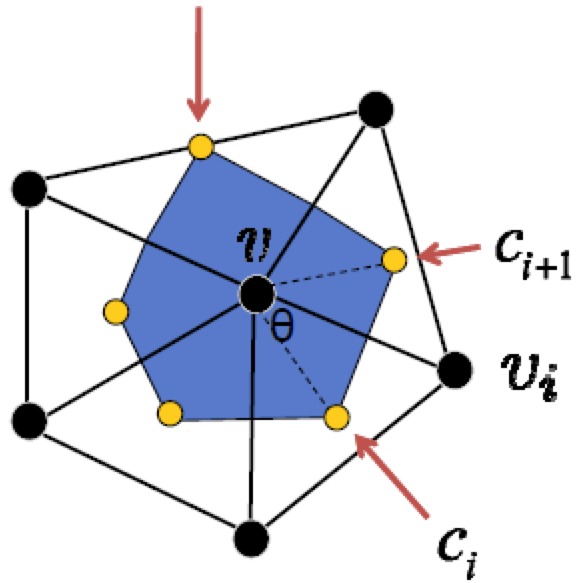
Mixed cell

http://graphics.stanford.edu/courses/cs468-12-spring/LectureSlides/05_Diff_Geo.pdf

Mixed Voronoi Cell

If $\theta < \pi/2$, c_i is the circumcenter of the triangle (v_i, v, v_{i+1})

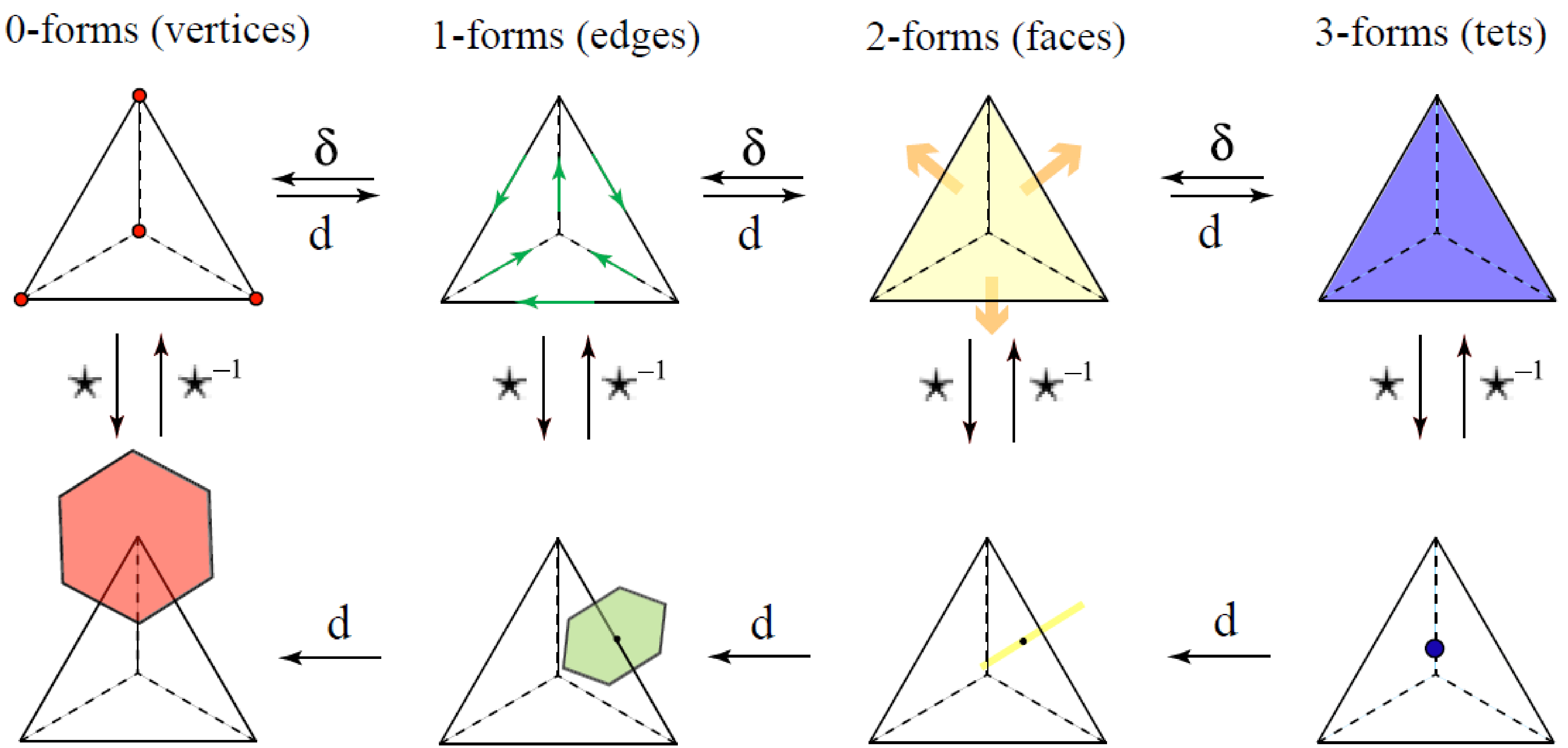
If $\theta \geq \pi/2$, c_i is the midpoint of the edge (v_i, v_{i+1})



$$A(v) = \sum_{v_i \in \mathcal{N}(v)} \left(Area(c_i, v, (v+v_i)/2) + Area(c_{i+1}, v, (v+v_i)/2) \right)$$

http://graphics.stanford.edu/courses/cs468-12-spring/LectureSlides/05_Diff_Geo.pdf

Discrete deRham Complex



http://ddg.cs.columbia.edu/SIGGRAPH06/DDGCourse2006.pdf

Co-Differential

$$\textsc{Theorem.}\quad \langle d\beta, \alpha \rangle = -\langle \beta, \star d \star \alpha \rangle$$

$$\boxed{\delta \equiv - \star d \star}$$

http://www.uib.no/filearchive/noteshodge27sep.pdf

Hodge Laplacian

$$\Delta = d \star d \star + \star d \star d$$

0-Form Laplacian

$$\Delta = \cancel{d\star d\star} + \star d \star d$$

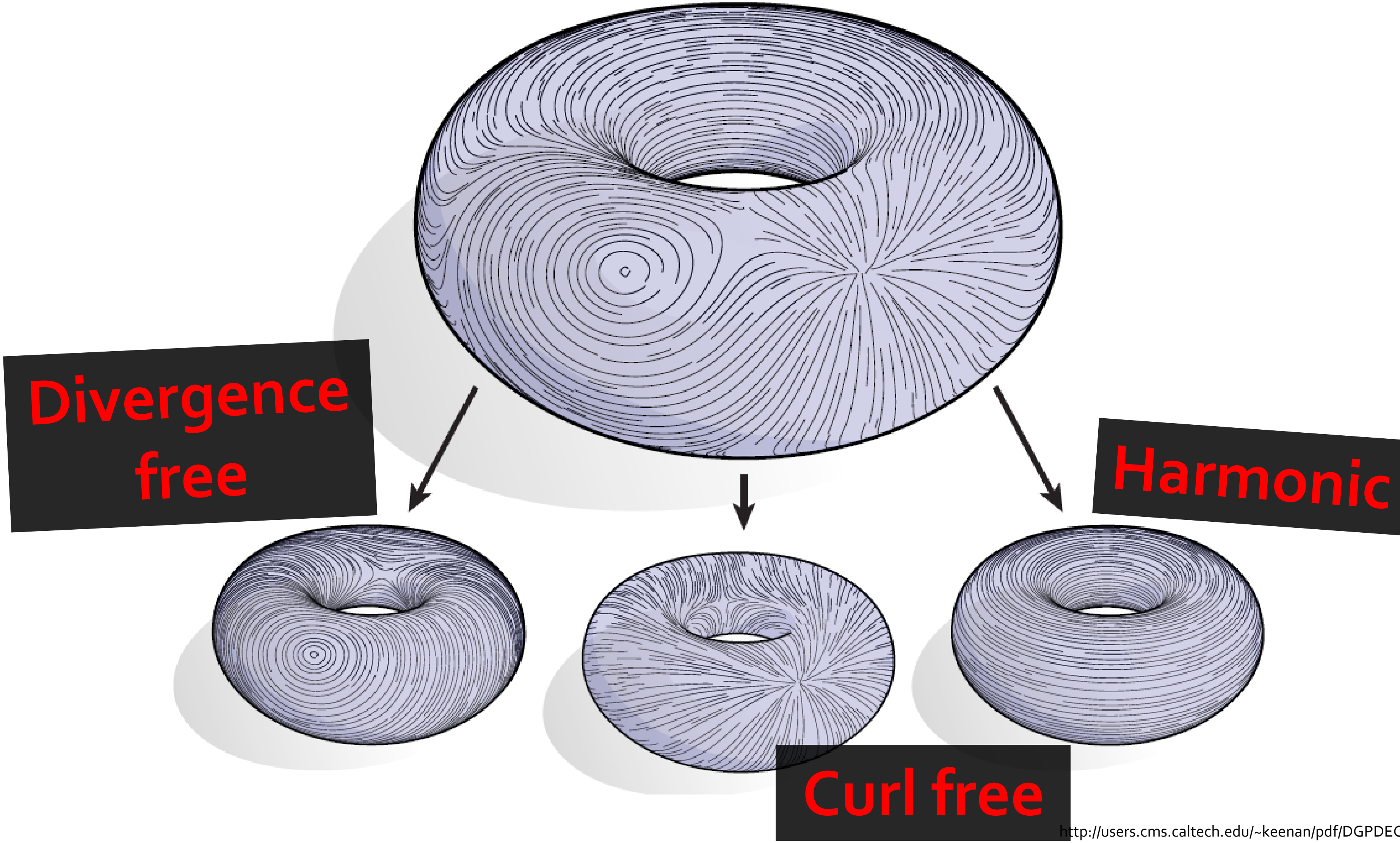
Cotangent Laplacian

0-Form Laplacian

$$\Delta = d \star d \star + \star d \star d$$

Area weights

Helmholtz-Hodge Decomposition

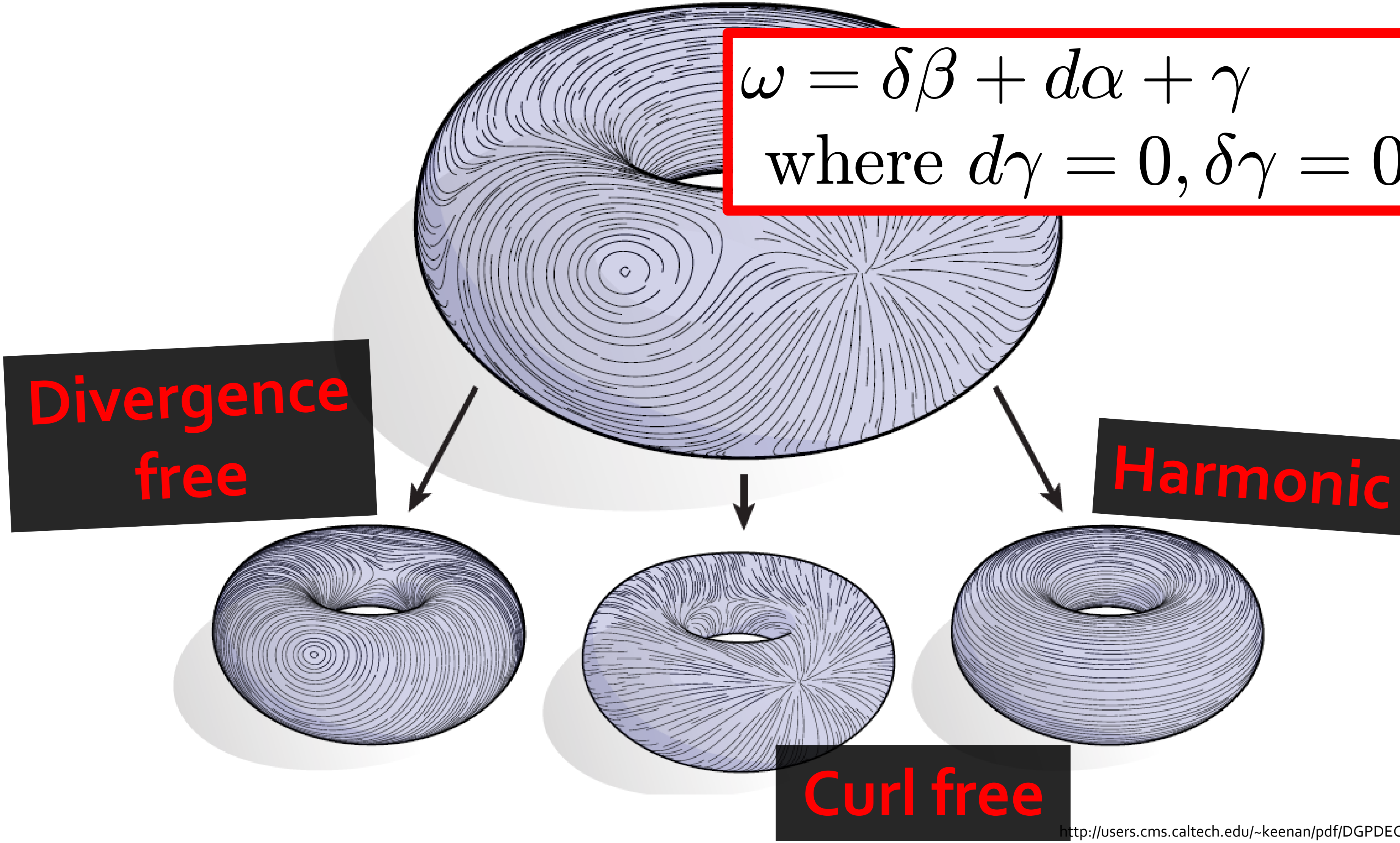


http://users.cms.caltech.edu/~keenan/pdf/DGPDEC.pdf

Helmholtz-Hodge Decomposition

$$\omega = \delta\beta + d\alpha + \gamma$$

$$\text{where } d\gamma = 0, \delta\gamma = 0$$



http://users.cms.caltech.edu/~keenan/pdf/DGPDEC.pdf

Computing the Decomposition

$$\omega = \delta\beta + d\alpha + \gamma$$
$$\text{where } d\gamma = 0, \delta\gamma = 0$$

Also exists a simple topological algorithm

$$\delta d\alpha = \delta\omega$$

$$d\delta\beta = d\omega$$

$$\gamma = \omega - \delta\beta - d\alpha$$

One-Form Laplacian Eigenforms

$$\omega = \delta\beta + d\alpha + \gamma$$
$$\text{where } d\gamma = 0, \delta\gamma = 0$$

$$\begin{aligned}\lambda(-\star d\bar{\beta} + d\alpha + \gamma) = \lambda\omega &= \Delta\omega \\ &= (d \star d \star + \star d \star d)(\delta\beta + d\alpha + \gamma) \\ &= (d \star d \star + \star d \star d)(-\star d \star \beta + d\alpha) \\ &= -\star d \star d \star d \star \beta + d \star d \star d\alpha \\ &= -\star d\Delta\bar{\beta} + d\Delta\alpha\end{aligned}$$

$$\bar{\beta} \equiv \star\beta$$

Conclusion: For $\lambda \neq 0$, they're obtained by d and $\star d$ of Laplacian eigenfunctions.

Recommended Reading

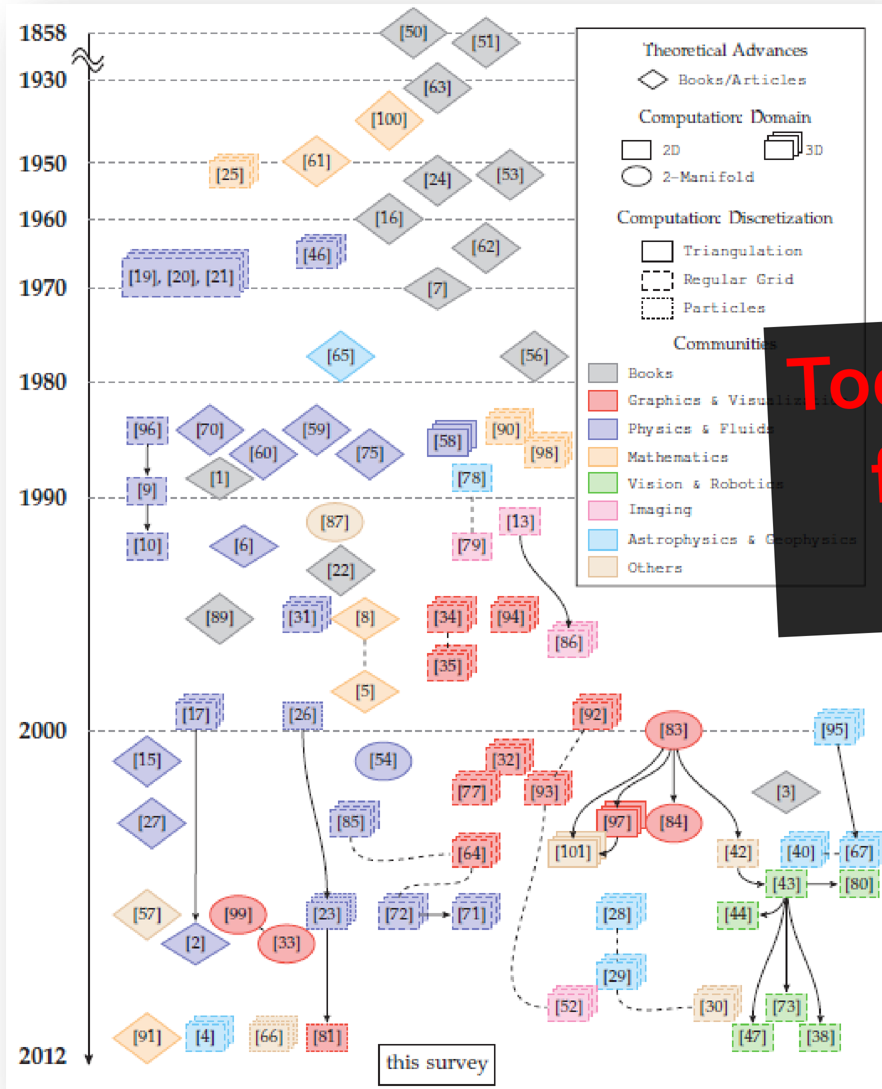
The Helmholtz-Hodge Decomposition - A Survey

Harsh Bhatia, *Student Member IEEE*, Gregory Norgard, Valerio Pascucci, *Member IEEE*, and Peer-Timo Bremer, *Member IEEE*

Abstract—The *Helmholtz-Hodge Decomposition (HHD)* describes the decomposition of a flow field into its divergence-free and curl-free components. Many researchers in various communities like weather modeling, oceanology, geophysics and computer graphics are interested in understanding the properties of flow representing physical phenomena such as incompressibility and vorticity. The HHD has proven to be an important tool in the analysis of fluids, making it one of the fundamental theorems in fluid dynamics. The recent advances in the area of flow analysis have led to the application of the HHD in a number of research communities such as flow visualization, topological analysis, imaging, and robotics. However, since the initial body of work, primarily in the physics communities, research on the topic has become fragmented with different communities working largely in isolation often repeating and sometimes contradicting each others results. Additionally, different nomenclature has evolved which further obscures the fundamental connections between fields making the transfer of knowledge difficult. This survey attempts to address these problems by collecting a comprehensive list of relevant references and examining them using a common terminology. A particular focus is the discussion of boundary conditions when computing the HHD. The goal is to promote further research in the field by creating a common repository of techniques to compute the HHD as well as a large collection of example applications in a broad range of areas.

Index Terms—Vector fields, Incompressibility, Boundary Conditions, Helmholtz-Hodge decomposition.

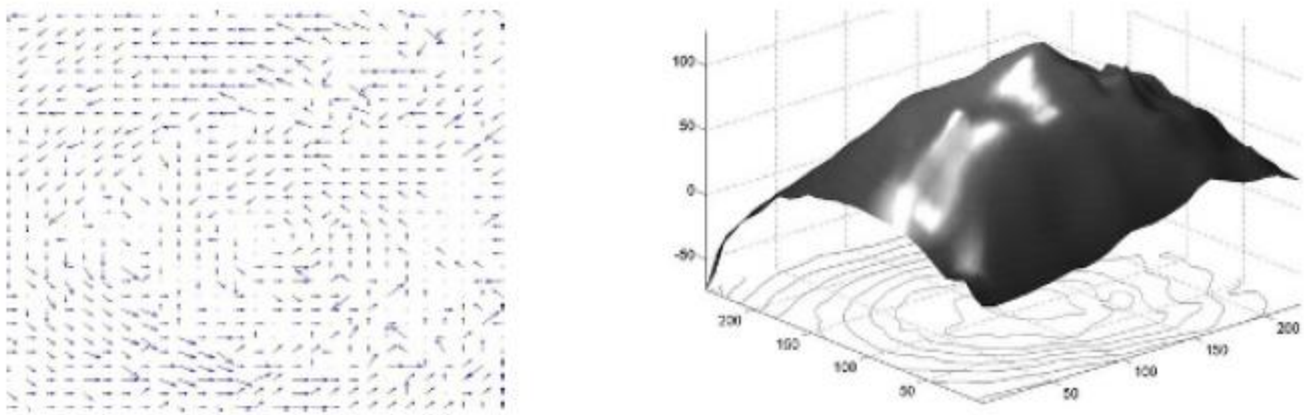
Recommended Reading



Today will take a few random samples

Simple Application

Fig. 2. Sequence of images from the Hurricane Luis sequence, with eye segmented



(a) (b)

Fig. 1. (a) Motion field in a anticlockwise rotating hurricane sequence extracted using the BMA. (b) The divergence free potential function with a distinct maximum and corresponding contours.

Palit, Basu, Mandal. "Applications of the Discrete Hodge Helmholtz Decomposition to Image and Video Processing." LNCS.

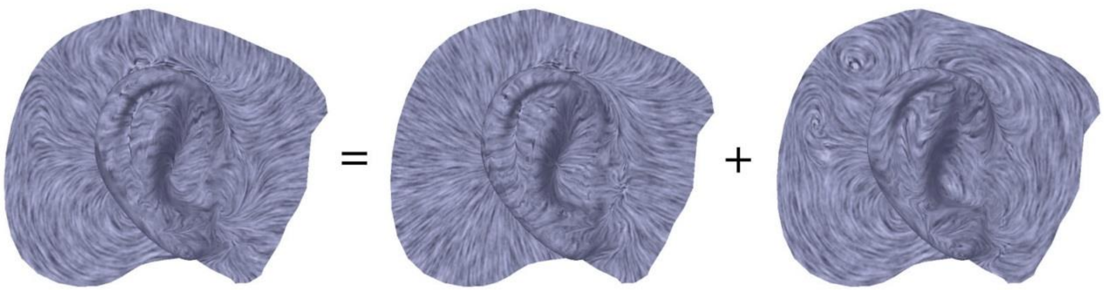
Fluid Simulation

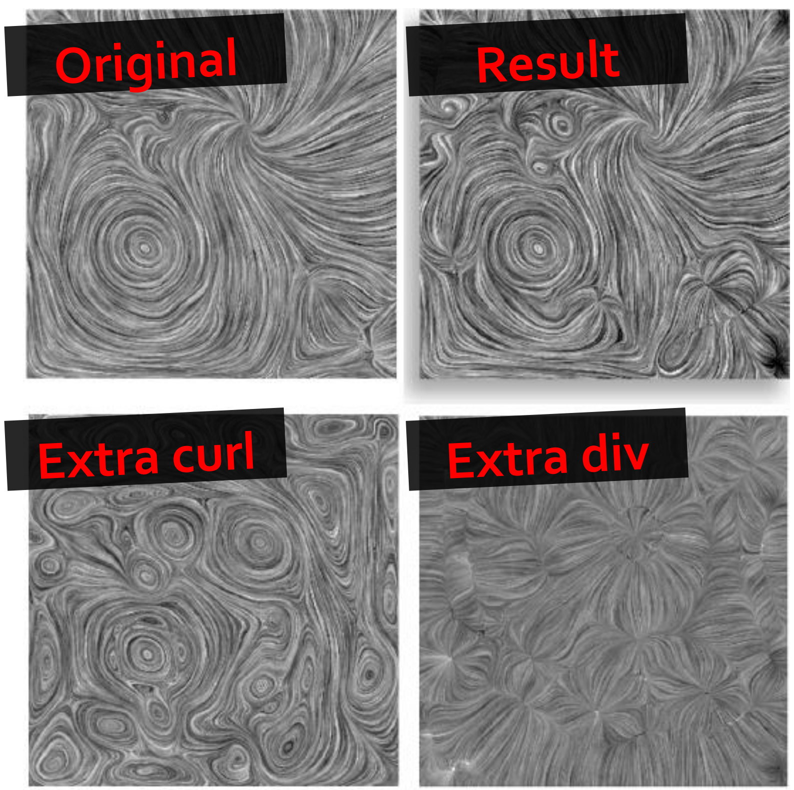
Divergence-free projection

Stam. "Stable Fluids." SIGGRAPH 1999. (and many others)

Incompressible: No divergence

Vector Field Editing

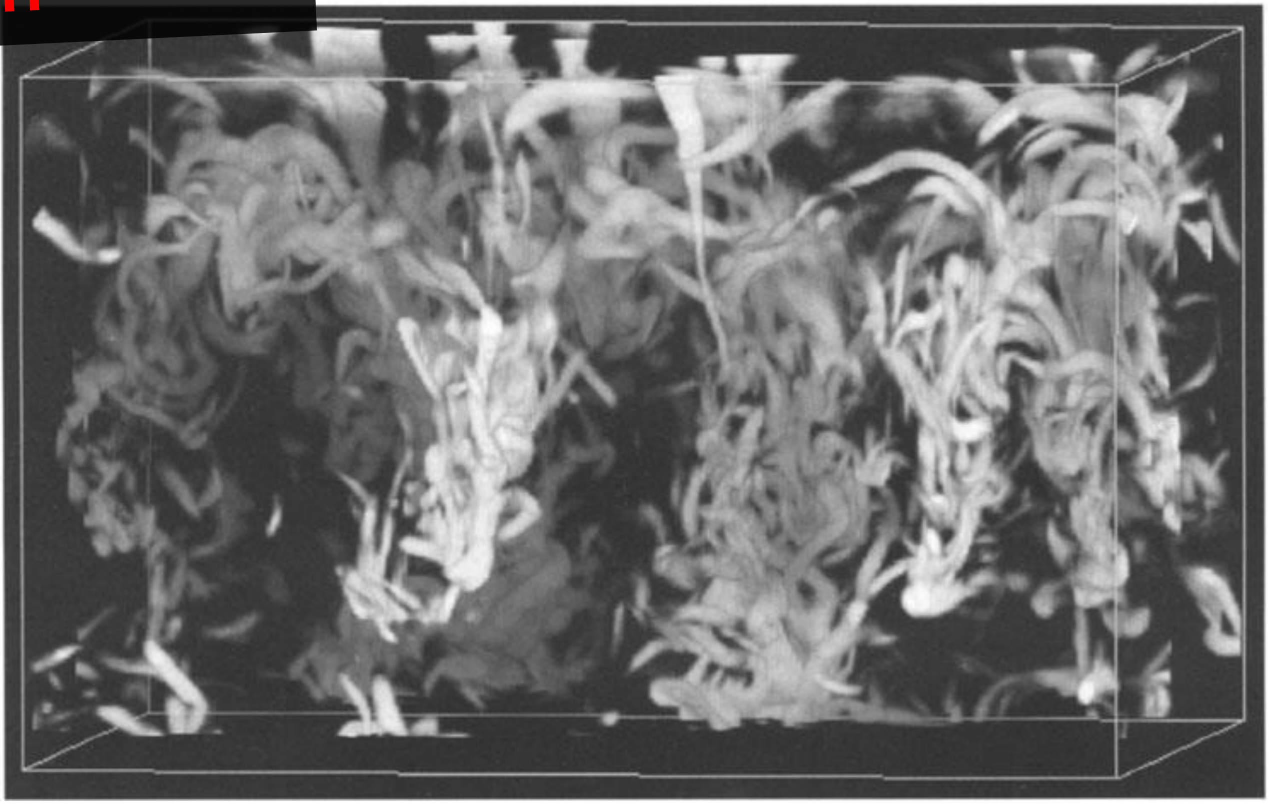




Tong et al. "Discrete Multiscale Vector Field Decomposition." TOG 2003.

Computational Physics

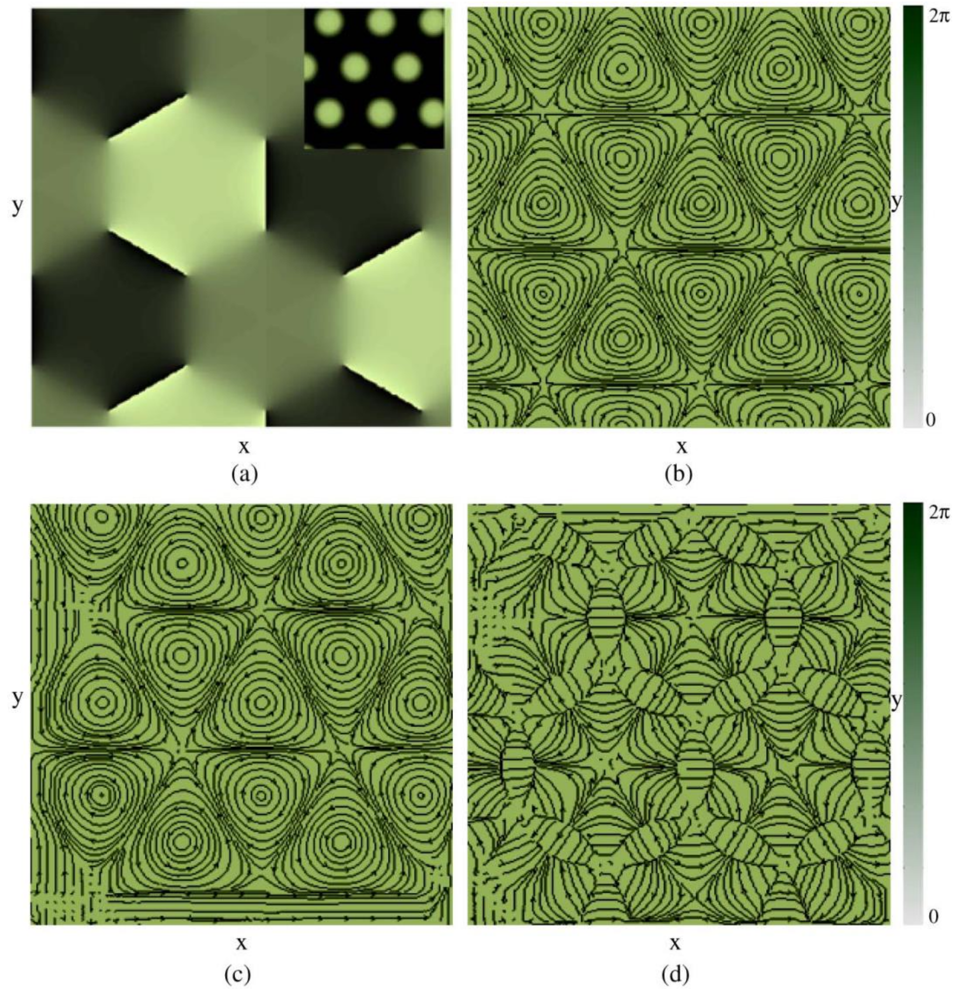
Separate turbulence from acoustics in solar simulation



Stein and Nordlund.
"Realistic Solar Convection Simulations."
***Solar Physics* 2000.**

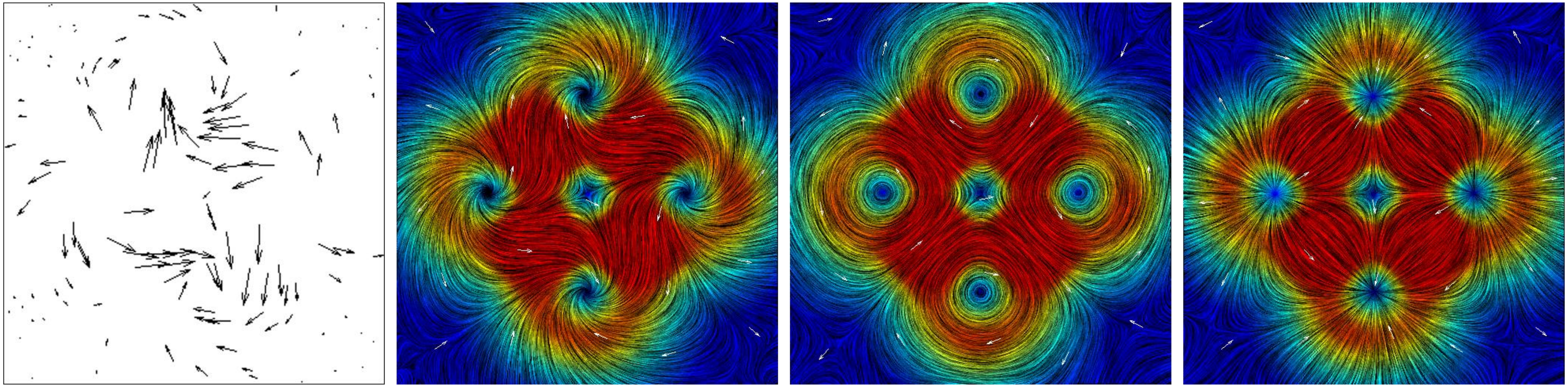
Computational Physics

Analyze interference and diffraction optics



Bahl and Senthilkumaran. "Helmholtz Hodge Decomposition of Scalar Optical Fields." J. Opt. Soc. Am. A 2012.

Reconstruct VF from Noisy Samples



$$\Phi_{df}(x) = H\phi(x) - tr\left\{H\phi(x)\right\}I$$

$$\Phi_{cf}(x) = -H\phi(x)$$

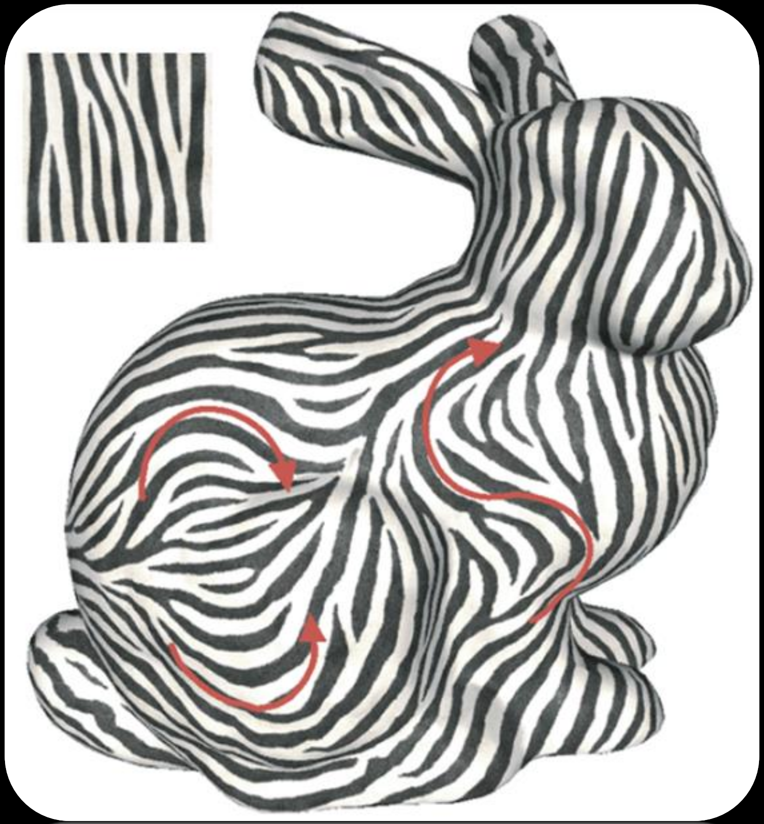
Macedo and Castro.
"Learning Divergence-Free and Curl-Free Vector Fields with Matrix-Valued Kernels."

Wrapping Up for Today

- Another cotangent Laplacian
- Helmholtz-Hodge Decomposition

Many more applications!

Fisher et al. "Design of Tangent Vector Fields" (SIGGRAPH 2007)



Discrete Exterior Calculus

CS 468, Spring 2013

Differential Geometry for Computer Science

Justin Solomon and Adrian Butscher