CS 468

Differential Geometry for Computer Science

Lecture 13 — Tensors and Exterior Calculus

Outline

- Linear and multilinear algebra with an inner product
- Tensor bundles over a surface
- Symmetric and alternating tensors
- Exterior calculus
- Stokes' Theorem
- Hodge Theorem

Inner Product Spaces

Let $\mathcal{V}$ be a vector space of dimension n.

Def: An inner product on $\mathcal{V}$ is a bilinear, symmetric, positive definite function $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$.

We have all the familiar constructions:

- The norm of a vector is $\|v\| := \sqrt{\langle v, v \rangle}$.
- Vectors v, w are orthogonal if $\langle v, w \rangle = 0$.
- If $\mathcal{S}$ is a subspace of $\mathcal{V}$ then every vector $v \in \mathcal{V}$ can be uniquely decomposed as $v := v^{\|} + v^{\perp}$ where $v^{\|} \in \mathcal{S}$ and $v^{\perp} \perp \mathcal{S}$.
- The mapping $v \mapsto v^{\|}$ is the orthogonal projection onto $\mathcal{S}$.

Dual Vectors

Def: Let $\mathcal{V}$ be vector space. The dual space is

$$\mathcal{V}^* := \{\xi : \mathcal{V} \to \mathbb{R} : \xi \text{ is linear}\}$$

Proposition: $\mathcal{V}^*$ is a vector space of dimension n.

Proof: If $\{E_i\}$ is a basis for $\mathcal{V}$ then $\{\omega^i\}$ is a basis for $\mathcal{V}^*$ where

$$\omega^i(E_s) = \begin{cases} 1 & i = s \\ 0 & \text{otherwise} \end{cases}$$

The Dual Space of an Inner Product Space

Let $\mathcal{V}$ be vector space with inner product $\langle \cdot, \cdot \rangle$. The following additional constructions are available to us.

- If $v \in \mathcal{V}$ then $v^\flat \in \mathcal{V}^*$ where $v^\flat(w) := \langle v, w \rangle \; \forall \, w \in \mathcal{V}$.
- If $\xi \in \mathcal{V}^*$ then $\exists \, \xi^\sharp \in \mathcal{V}$ so that $\xi(w) = \langle \xi^\sharp, w \rangle \; \forall \, w \in \mathcal{V}$.
- These are inverse operations: $(v^\flat)^\sharp = v$ and $(\xi^\sharp)^\flat = \xi$.
- $\mathcal{V}^*$ carries the inner product $\langle \xi, \zeta \rangle_{\mathcal{V}^*} := \langle \xi^\sharp, \zeta^\sharp \rangle \; \forall \, \xi, \zeta \in \mathcal{V}^*$

Basis Representations

Let $\{E_i\}$ denote a basis for $\mathcal{V}$ and put $g_{ij} := \langle E_i, E_j \rangle$.

Def: Let g^{ij} be the components of the inverse of the matrix $[g_{ij}]$.

Then:

- The dual basis is $\omega^i := \sum_j g^{ij} E_j$.
- If $v = \sum_i v^i E_i$ then $v^\flat = \sum_i v_i \omega^i$ where $v_i := \sum_j g_{ij} v^j$.
- If $\xi = \sum_i f_i \eta^i$ then $f^\sharp = \sum_i f^i E_i$ where $f^i := \sum_j g^{ij} f_j$.
- If $\xi = \sum_i a_i \omega^i$ and $\zeta = \sum_i b_i \omega^i$ then $\langle \xi, \zeta \rangle = \sum_{ij} g^{ij} a_i b_j$

Note: If $\{E_i\}$ is orthonormal then $g_{ij} = \delta_{ij}$ and $v_i = v^i$ and $\xi^i = \xi_i$.

Tensors

Let $\mathcal{V}$ be a vector space of dimension n.

Tensors are "multilinear functions on $\mathcal{V}$ with multi-vector output."

Def: The space of k-covariant and ℓ-contravariant tensors is

$$\mathcal{V}^* \overbrace{\otimes \cdots \otimes}^{k \text{ times}} \mathcal{V}^* \otimes \mathcal{V} \overbrace{\otimes \cdots \otimes}^{\ell \text{ times}} \mathcal{V} := \left\{ \begin{array}{c} f : \mathcal{V} \overbrace{\times \cdots \times}^{k \text{ times}} \mathcal{V} \to \mathcal{V} \overbrace{\times \cdots \times}^{\ell \text{ times}} \mathcal{V} \\ \text{such that } f \text{ is multilinear} \end{array} \right\}$$

Basic facts:

- Vector space of dimension $n^{k+\ell}$. Basis in terms of E_i's and ω^i's.
- Inherits an inner product from $\mathcal{V}$ and has $\sharp$ and $\flat$ operators.
- There are contractions (killing a $\mathcal{V}$ factor with a $\mathcal{V}^*$ factor).

Symmetric Bilinear Tensors

A symmetric (2,0)-tensor is an element $A \in \mathcal{V}^* \otimes \mathcal{V}^*$ such that $A(v, w) = A(w, v)$ for all $v, w \in \mathcal{V}$.

Example:

$A = 2^{nd}$ FF and
$S =$ shape operator.

Some properties:

- In a basis we have $A = \sum_{ij} A_{ij}\, \omega^i \otimes \omega^j$ with $A_{ij} = A_{ji}$.
- We define an associated self-adjoint (1,1)-tensor $S \in \mathcal{V}^* \otimes \mathcal{V}$ with the formula $A(v, w) := \langle S(v), w \rangle$.
- In a basis we have $S = \sum_{ij} S_i^j \omega^i \otimes E_j$ where $S_i^j = \sum_k g^{kj} A_{ik}$.
- If $v = \sum_i v^i E_i$ and $w = \sum_i w^i E_i$ then $\langle v, w \rangle = [v]^\top [g][w]$ and

$$A(v, w) = [v]^\top [A][w] \qquad \text{and} \qquad S = [g]^{-1}[A]$$

- The contraction of A equals the trace of S equals $\sum_{ij} g^{ij} A_{ij}$

Alternating Tensors

A *k-form* is an element $\sigma \in \mathcal{V}^* \otimes \cdots \otimes \mathcal{V}^*$ such that for all $v, w \in \mathcal{V}$ and pairs of slots in σ we have

$$\sigma(\dots v \dots w \dots) = -\sigma(\dots w \dots v \dots)$$

"Alternating $(k, 0)$-tensor"

Fact: If $\dim \mathcal{V} = 2$ then only $k = 0, 1, 2$ are non-trivial.

$$\mathrm{Alt}^0(\mathcal{V}) = \mathbb{R} \quad \text{and} \quad \mathrm{Alt}^1(\mathcal{V}) = \mathcal{V}^* \quad \text{and} \quad \mathrm{Alt}^2(\mathcal{V}) \cong \mathbb{R}$$

Basis: The element $\omega^1 \wedge \omega^2$

Let $v = \sum_i v^i E_i$ and $w = \sum_i w^i E_i$. Then we define it via

$$\omega^1 \wedge \omega^2(v, w) := \det([v\ w])$$

Duality: if $\mathcal{V}$ has an inner product

- The *area form* $dA \in \mathrm{Alt}^2(\mathcal{V})$

$$dA(v, w) := \left[\begin{array}{c} \text{Signed area of} \\ \text{parallelogon } v \wedge w \end{array} \right]$$

- The *Hodge-star* operator $*$

$$\omega \wedge *\tau := \langle \omega, \tau \rangle \, dA$$

$*dA = 1$ and if $\omega \in \mathrm{Alt}^1(\mathcal{V})$ then

$*1 = dA$ $\quad *\omega(v) = \omega(R_{\pi/2}(v))$

Tensor Bundles on a Surface

Let S be a surface and let $\mathcal{V}_p := T_pS$.

Def: The bundle of (k, ℓ)-tensors over S attached the vector space $\mathcal{V}_p^{(k,\ell)} := \mathcal{V}_p^* \otimes \cdots \otimes \mathcal{V}_p^* \otimes \mathcal{V}_p \otimes \cdots \otimes \mathcal{V}_p$ at each $p \in S$.

Def: A section of this bundle is the assignment $p \mapsto \sigma_p \in \mathcal{V}_p^{(k,\ell)}$.

Examples:

- $k = \ell = 0$ — sections are functions on S
- $k = 0, \ell = 1$ — sections are vector fields on S
- $k = 1, \ell = 0$ — sections are one-forms on S
- $k = 2, \ell = 0$ and symmetric — sections are a symmetric bilinear form at each point. E.g. the metric and the 2^{nd} FF.
- $k = 2, \ell = 0$ and antisymmetric — sections are two-forms on S. E.g. the area form.

Covariant Differentiation in a Tensor Bundle

The covariant derivative extends naturally to tensor bundles.

A formula: Choose a basis and suppose

$$\sigma := \sum_{ijkl} \sigma_{ij}^{k\ell} \omega^i \otimes \omega^j \otimes E_k \otimes E_\ell$$

is a tensor. Then

$$\nabla\sigma := \sum_{ijkls} \nabla_s \sigma_{ij}^{k\ell} [\omega^i \otimes \omega^j \otimes E_k \otimes E_\ell] \otimes \omega_s$$

is also a tensor, where

$$\nabla_s \sigma_{ij}^{k\ell} := \frac{\partial \sigma_{ij}^{k\ell}}{\partial x^s} - \Gamma_{is}^t \sigma_{tj}^{k\ell} - \Gamma_{js}^t \sigma_{it}^{k\ell} + \Gamma_{ts}^i \sigma_{ij}^{t\ell} + \Gamma_{ts}^\ell \sigma_{ij}^{kt}$$

Exterior Differentiation

Def: The exterior derivative is the operator $d : \mathrm{Alt}^k(S) \to \mathrm{Alt}^{k+1}(S)$ defined as follows.

- Choose a basis.
- If $f \in \mathrm{Alt}^0(S)$ we define df geometrically by $df(V) := V(f)$ or

$$df = \sum_i \frac{\partial f}{\partial x^i}\omega^i \qquad \text{Thus } (df)^\sharp = \nabla f$$

- If $\omega = \sum_i a_i \omega^i \in \mathrm{Alt}^1(S)$ then $d\omega = \left(\frac{\partial a^1}{\partial x^2} - \frac{\partial a^2}{\partial x^1}\right)\omega^1 \wedge \omega^2$
- If $\omega = a\,\omega^1 \wedge \omega^2 \in \mathrm{Alt}^0(S)$ then $d\omega = 0$.

Basic Facts:

- $dd\omega = 0$ for all $\omega \in \mathrm{Alt}^k(S)$ and all k.
- $d\omega = \mathrm{Antisym}(\nabla\omega)$.

The Co-differential Operator

Def: The co-differential is the L^2-adjoint of d. It is therefore an operator $\delta : \mathrm{Alt}^{k+1}(S) \to \mathrm{Alt}^k(S)$ that satisfies

$$\int_S \langle d\omega, \tau \rangle \, dA = \int_S \langle \omega, \delta\tau \rangle \, dA$$

It is given by $\delta := - * d *$.

Interpretations:

- If f is a function, then $(df)^\sharp = \nabla f$.
- If X is a vector field, then $\delta X^\flat = div(X)$.
- If X is a vector field, then $dX^\flat = curl(X) \, dA$.
- If f is a function, then $\big(\delta(f \, dA)\big)^\flat = R_{\pi/2}\big(\nabla f\big)$.

Stokes' Theorem

Intuition: Generalization of the Fundamental Theorem of Calculus.

Suppose that c be a $(k+1)$-dimensional submanifold of S with k-dimensional boundary ∂c. Let ω be a k-form on S. Then:

$$\int_c d\omega = \int_{\partial c} \omega$$

Interpretations:

- The divergence theorem:

$$\int_S div(X) dA = \int_{\partial S} \langle N_{\partial S}, X \rangle \, d\ell$$

- Etc.

The Hodge Theorem

Theorem: $\mathrm{Alt}^1(S) = d\mathrm{Alt}^0(S) \oplus \delta \mathrm{Alt}^2(S) \oplus \mathcal{H}^1$ where $\mathcal{H}^1$ is the set of harmonic one-forms:

$$h \in \mathcal{H}^1 \quad \Leftrightarrow \quad dh = 0 \text{ and } \delta h = 0$$

$$\Leftrightarrow \quad \underbrace{(d\delta + \delta d)}_{\text{“Hodge Laplacian”}} h = 0$$

Corollary: Every vector field X on S can be decomposed into a “gradient” part, a “divergence-free” part, and a “harmonic part.”

$$X = \nabla \phi + curl(\nabla \psi) + h^\sharp \quad \text{with } h \in \mathcal{H}^1$$

Another deep mathematical result:

Theorem: $\dim(\mathcal{H}^1) = 2\chi(S)$. This is a toplogical invariant.