Homework 4: DEC and Curvature

Differential Geometry for Computer Science (Spring 2013), Stanford University

Due **Wednesday**, June 5, in the course mailbox

Problem 1 (30 points)**.**

(a) We defined the coefficients of the Riemann curvature $(3,1)$-tensor with respect to the coordinate basis E_1, E_2 by

$$\sum_s R_{ijk}{}^s E_s := \nabla_{E_j} \nabla_{E_i} E_k - \nabla_{E_i} \nabla_{E_j} E_k \,.$$

Derive the formula

$$R_{ijk}{}^s = \frac{\partial \Gamma^s_{jk}}{\partial x^i} - \frac{\partial \Gamma^s_{ik}}{\partial x^j} + \sum_t \Gamma^t_{jk} \Gamma^s_{it} - \sum_t \Gamma^t_{ik} \Gamma^s_{jt} \,.$$

(b) We defined the coefficients of the Riemann curvature $(4,0)$-tensor by $R_{ijk\ell} := \sum_s g_{\ell s} R_{ijk}{}^s$ or equivalently by $R_{ijk\ell} := g(\nabla_{E_j} \nabla_{E_i} E_k - \nabla_{E_i} \nabla_{E_j} E_k, E_\ell)$. Use Gauss' Theorema Egregium to verify the so-called symmetries *of the curvature tensor:*

$$R_{ijk\ell} = -R_{jik\ell} \qquad R_{ijk\ell} = -R_{ij\ell k} \qquad R_{ijk\ell} = R_{k\ell ij} \,.$$

(c) Show that on a 2-dimensional surface, the only independent component of the Riemann curvature $(4,0)$-tensor is R_{1212}. In other words, show that all other components of Rm *are either zero or a multiple of R_{1212}.*

(d) Use intrinsic calculations to find the Riemann curvature $(4,0)$-tensor of the sphere. (Hints: you get to choose the parametrization of the sphere — so choose wisely; also you only need to compute R_{1212}!)

(e) Find the Gauss curvature of the sphere via the second fundamental form. Compare with part (d) and verify Gauss' Theorema Egregium.

Problem 2 (20 points)**.** *Differential geometry is all about finding good local coordinate systems for a surface S which then help prove theorems. For instance, the Gauss-Bonnet theorem uses an* orthogonal parametrization*. This is a parametrization $\phi : \mathcal{U} \subseteq \mathbb{R}^2 \to S$ with the property that $g_{12}(x) = 0$ for all $x \in \mathcal{U}$. In other words, if $E_i := \frac{\partial \phi}{\partial x^i}$ then $\langle E_1, E_2 \rangle = 0$ at all points on S in the image of ϕ. (In this coordinate system, it is not necessarily the case that $\langle E_1, E_1 \rangle = \langle E_2, E_2 \rangle = 1$. In fact, if this were to hold, then S would have a neighbourhood that is isometric to Euclidean space, which can happen if and only if the Riemann curvature tensor of S is zero inside $\mathcal{U}$.)*

Suppose that $\gamma : [-1,1] \to S$ is a geodesic segment in S. For every $s \in [0,1]$, let $N(s)$ be the unit vector in $T_{\gamma(s)}S$ that is orthogonal to $\dot\gamma(s)$. In this problem, you will prove that the mapping $\phi(s,t) := \exp_{\gamma(s)}(tN(s))$ for $s \in (-1,1)$ and small t is an orthogonal parametrization of a neighbourhood of $\gamma(0)$. In fact, you will do slightly better and show that $g_{12} = 0$ and $g_{22} = 1$ for all (s,t) in the parameter domain.

(a) *Draw an informative picture. What could go wrong if t is allowed to become too large?*

(b) *Let $E_1 := \frac{\partial \phi}{\partial s}$ and $E_2 := \frac{\partial \phi}{\partial t}$. Show that $\nabla_{E_2} E_2 = 0$ for all s, t.*

(c) *Show that $\|E_2\| = 1$ for all (s,t). (Hint: why is this true when s is arbitrary and $t = 0$? Now hold s fixed and show that $\frac{\partial}{\partial t}\|E_2\|^2 = 0$ for all t.) Conclude that $g_{22} = 1$ for all (s,t).*

(d) *Show that $\langle E_1, E_2 \rangle = 0$ for all (s,t). (Hint: why is this true when s is arbitrary and $t = 0$? Now hold s fixed and show that $\frac{\partial}{\partial t}\langle E_1, E_2 \rangle = 0$ for all t.) Conclude that $g_{12} = 0$ for all (s,t).*

Problem 3 (20 points). *The* divergence theorem *says that for any smooth vector field X on a surface S with boundary ∂S, we have*

$$\int_S \operatorname{div}(X)\, dA = \int_{\partial S} \langle X, N \rangle\, ds\,.$$

where dA is the Riemannian area form, N is a unit vector tangent to S but normal to ∂S, and we must use an arc-length parametrization for ∂S for this equation to hold. Stokes' Theorem says that for any differential k-form ω and $(k+1)$-dimensional submanifold $c \subseteq S$ we have

$$\int_{\partial c} \omega = \int_c d\omega\,.$$

In this problem, you will show that Stokes' Theorem implies the divergence theorem for a well-chosen ω. This is a straightforward problem that the unfamiliar notation of differential forms and sharp/flat/star operators may make quite difficult. Do your best!

(a) *Show that $\operatorname{div}(X) = - * d * (X^\flat)$. Hint: you need to show this at an arbitrary point $p \in S$ using your favourite coordinate system. So work in geodesic normal coordinates centered at p.*

(b) *Explain why $\operatorname{div}(X) dA$ can be put in the form $d\omega$ for some form ω, and what is ω?*

(c) *Apply Stokes' Theorem to $d\omega$ and S itself. We thus get $\int_S \operatorname{div}(X) dA = \int_{\partial S} \omega$. To develop the right hand side further, you must know how to evaluate the "line integral" $\int_{\partial S} \omega$. Suppose that we can parametrize the boundary ∂S by arc-length as a curve $\gamma : [0, \ell] \to S$ with tangent vector $T(s) := \dot{\gamma}(s)$. Now $\int_{\partial S} \omega$ is defined to be $\int_0^\ell \omega(T(s)) ds$. Show that $\omega(T) = \langle X, N \rangle$ where N is the vector obtained by rotating T counterclockwise by $\pi/2$.*

Problem 4 (15 points). *Recall that the Helmholtz-Hodge decomposition of a one-form ω is given by $\omega = \delta\beta + d\alpha + \gamma$, where $d\gamma = 0$ and $\delta\gamma = 0$. In lecture we argued that you can find the Helmholtz-Hodge decomposition in DEC by solving $\delta d\alpha = \delta\omega$ and $d\delta\beta = d\omega$ (and taking $\gamma = \omega - \delta\beta - d\alpha$).*

(a) *Argue that the operators δd and $d\delta$ have null spaces for closed triangulated surfaces. Why isn't this a hole in our technique?*

(b) *Compute `helmholtzHodge.m` implementing this technique and visualize the results using `problem4.m`. Notice that we have kindly provided `discreteExteriorCalculus.m` implementing the DEC matrices you will need.*

Problem 5 (15 points). *As promised, we return to the problem of geodesic computation:*

(a) *When does the planar front approximation made in the fast marching algorithm behave well? When does it behave poorly?*

(b) *In 2002, Novotni and Klein proposed using a* circular *wavefront rather than a planar wavefront in fast marching. For the most part, the algorithm remains the same, since it is a simple extension of Dijkstra's algorithm for shortest paths, but the update step must be changed. Without loss of generality, we'll embed three vertices of a triangle being updated onto the plane at positions* $v_1 \equiv 0$, $v_2 \equiv (v_{2x}, 0)$, *and* $v_3 = (v_{3x}, v_{3y})$ *with* $v_{3y} \geq 0$ *(make sure you understand why such an embedding is possible); we know distances* d_1 *and* d_2 *but want to find* d_3.

   (i) *Given* d_1 *and* d_2, *write and solve a system of equations for finding the source point* (x, y) *of the circular wavefront.*

   (ii) *Your system from (i) should be quadratic and thus yields two solutions. Provide a rule for choosing one of the two roots to give a single point* (x, y), *and give an expression for* d_3.