Direct Construction of the Approximate Voronoi Diagram

Primoz Skraba

October 30, 2006

CS 486

Well-Separatedness

- Definition

  X and Y are *well separated* if they can be enclosed withing two disjoint d-dimensional balls of radius r, such that the distance between the centers of the balls is at least αr

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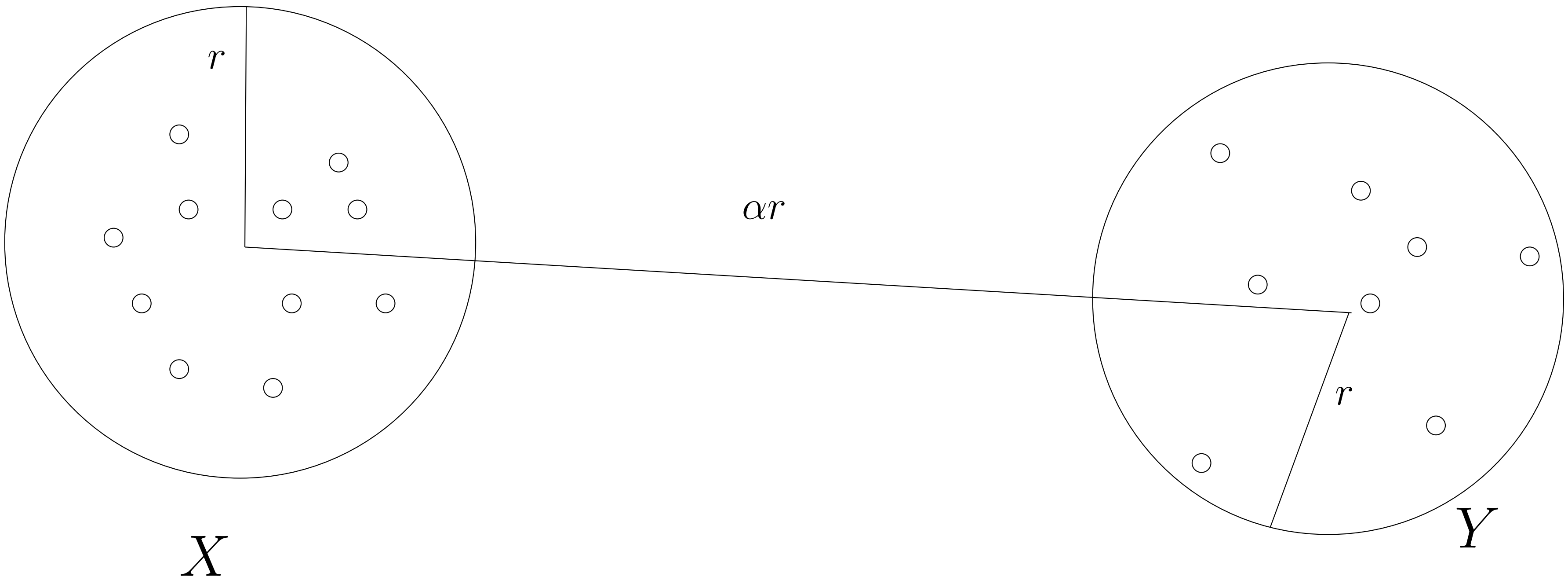
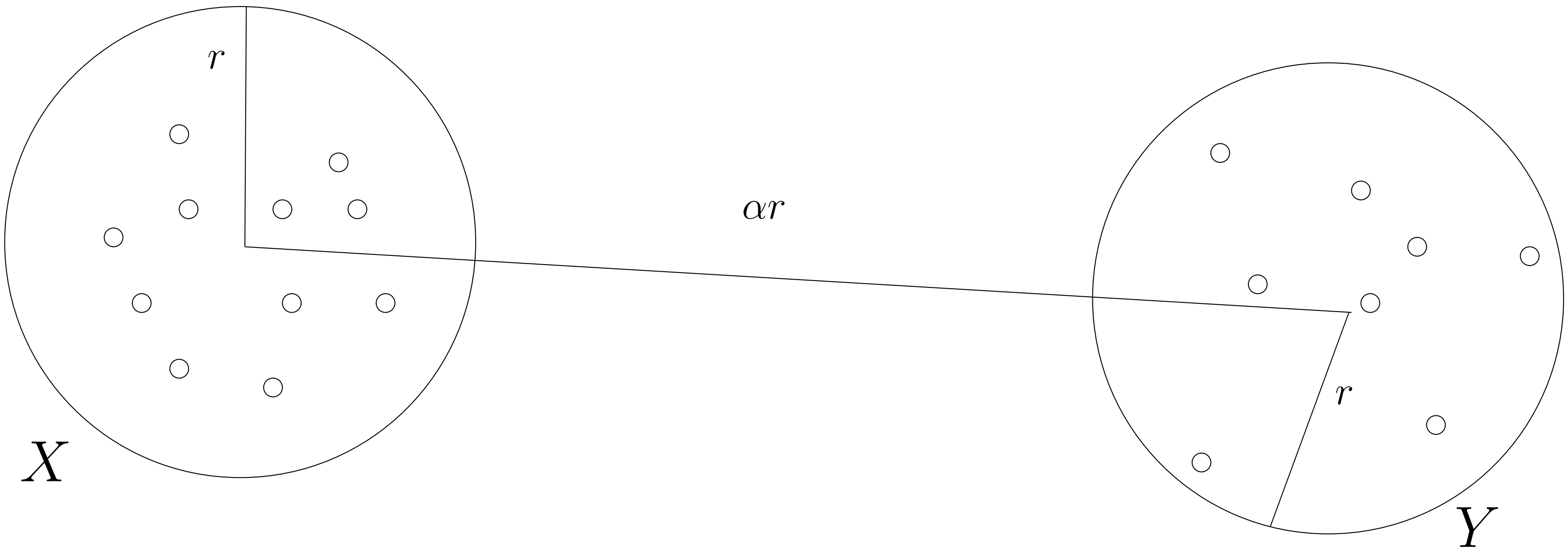


Well-Separated Pair Decomposition

- Definition

  A *well-separated pair decomposition (WSPD)* is a set $P_{S,\alpha} = \{(X_1, Y_1), \ldots, (X_m, Y_m)\}$ of pairs of subset so that each pair is well-separated and for any two distinct points $x, y \in S$ there exists a pair (X_i, Y_i) which separates them.



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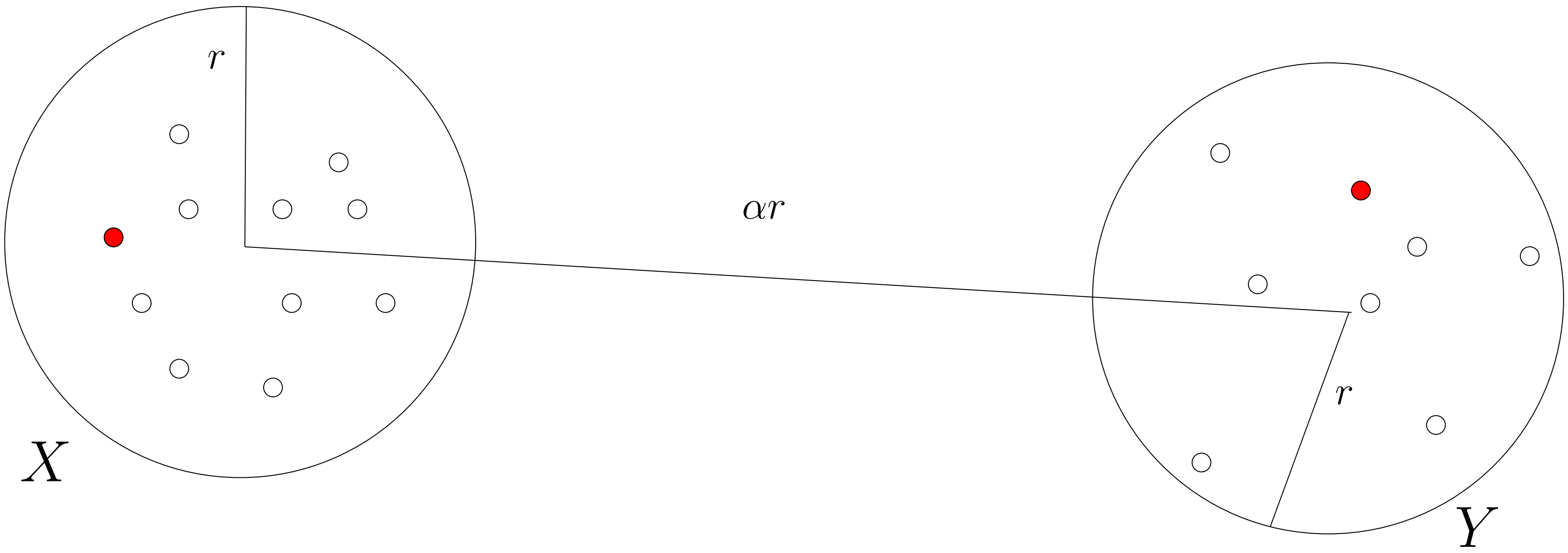

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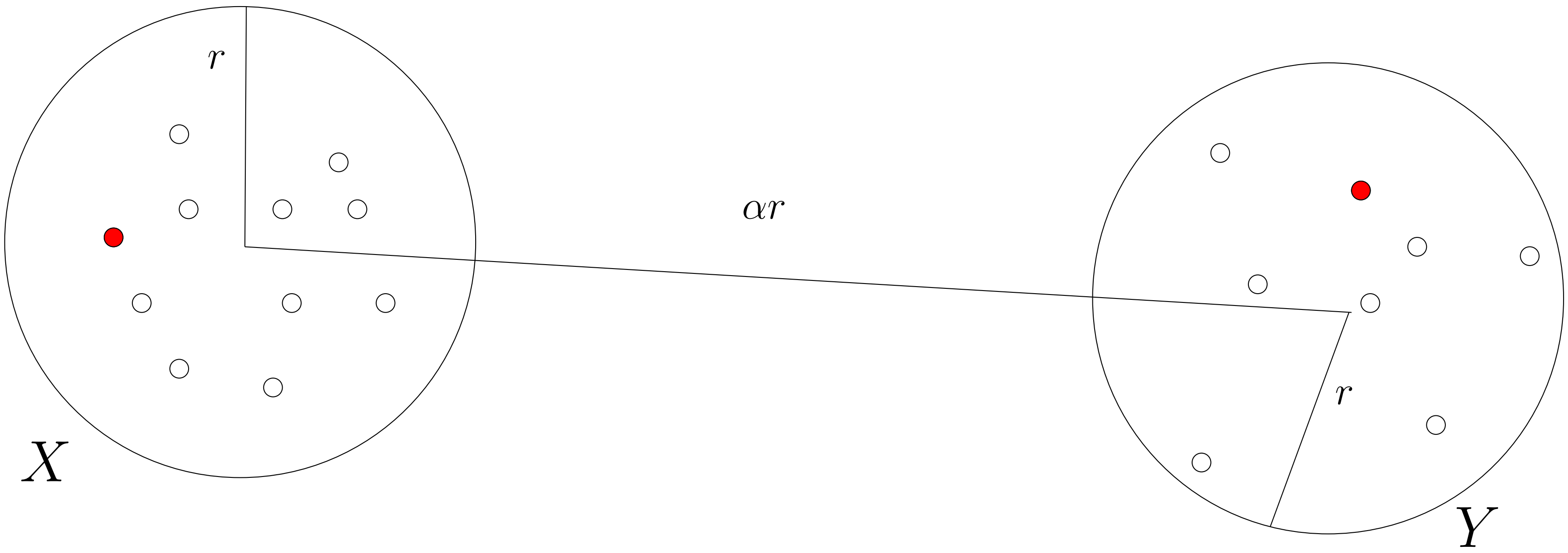

- Properties
  - Can be constructed in $O(n \log n + \alpha^d n)$ time
  - Contains $O(\alpha^d n)$ pairs

Constructing a WSPD

- Due to Callahan and Kosaraju (Fair-Split Tree)
- Can use a quadtree
- **Algorithm**
  - Take cubes (u, v), if they are well separated, add the pair and terminate
  - If not, call function on (w, v) where w are the children of u

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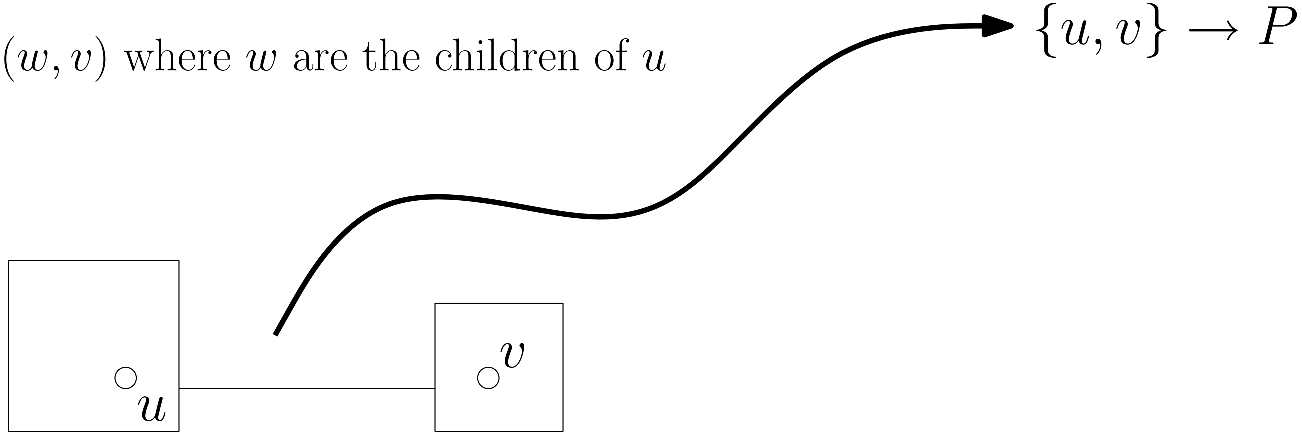

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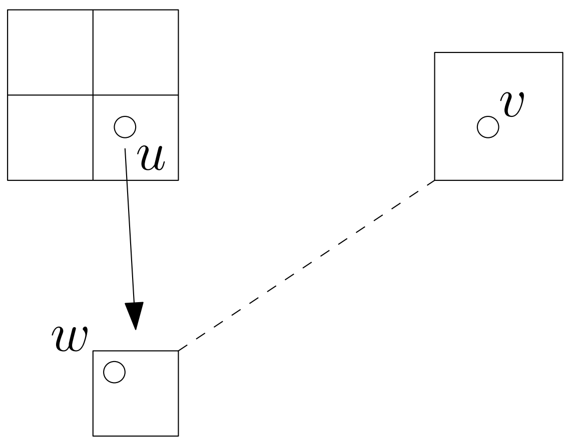

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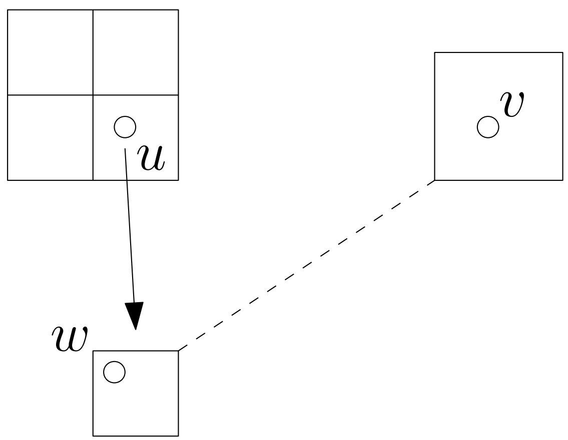
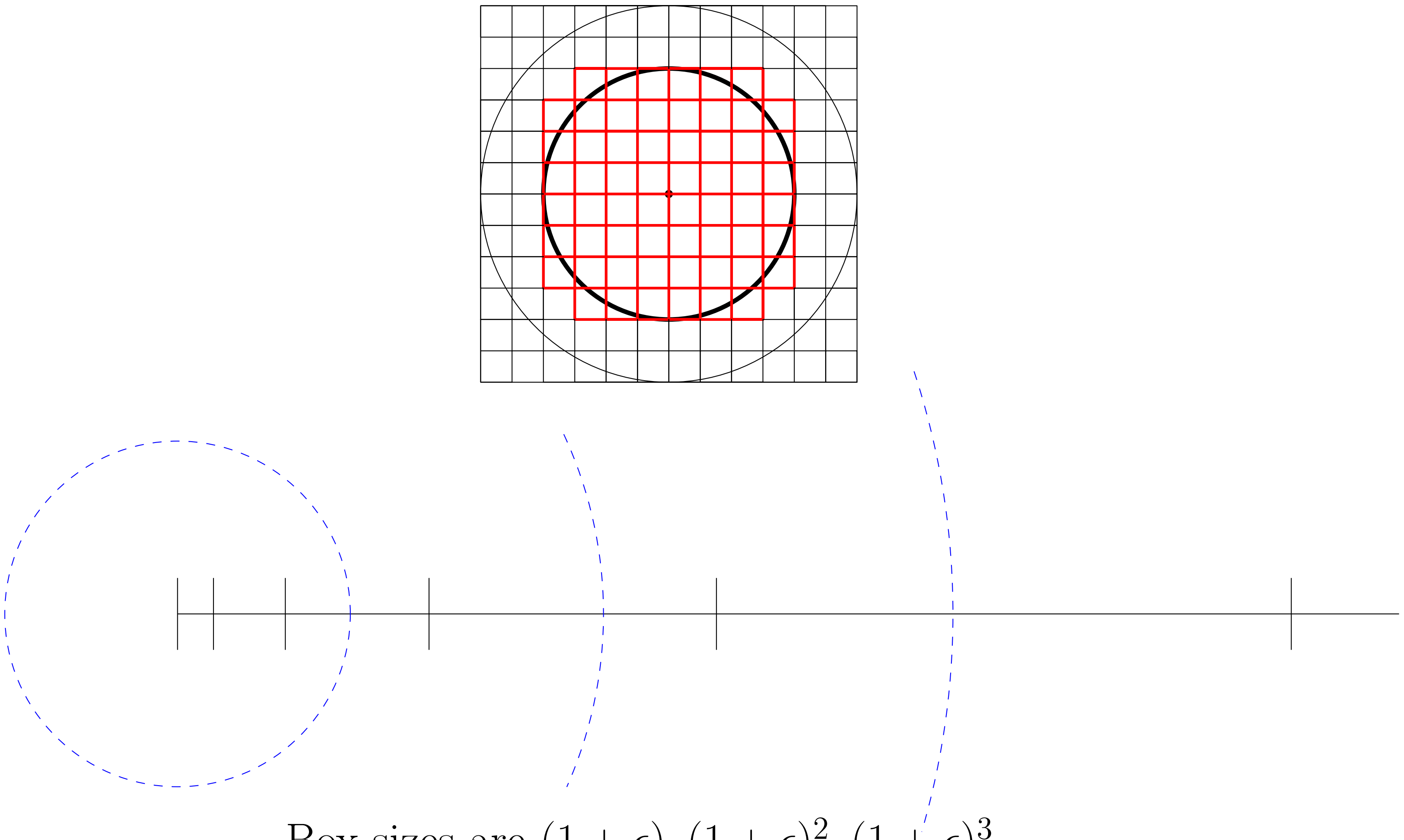


Always take the children of the larger cell!

Number of cells in a ball

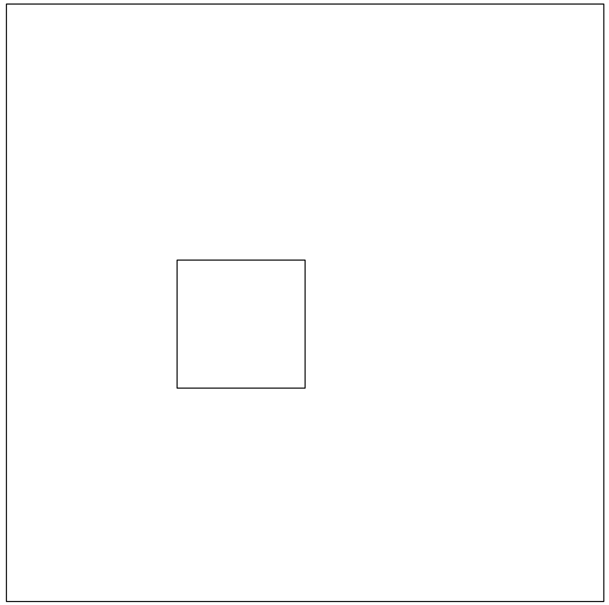
- Place grid cells around point to fill up the ball of radius r



Box sizes are $(1+\epsilon), (1+\epsilon)^2, (1+\epsilon)^3, \ldots$

Number of cells is $O(\frac{1}{\epsilon^d})$

Balanced Box Decomposition Tree



- Definition

  Each cell is the difference between and *inner quadtree box* and *outer quadtree box*

- Properties

  For any collection C of quadtree boxes

  1. with $O(|C|)$ nodes
  2. $O(\log |C|)$ depth
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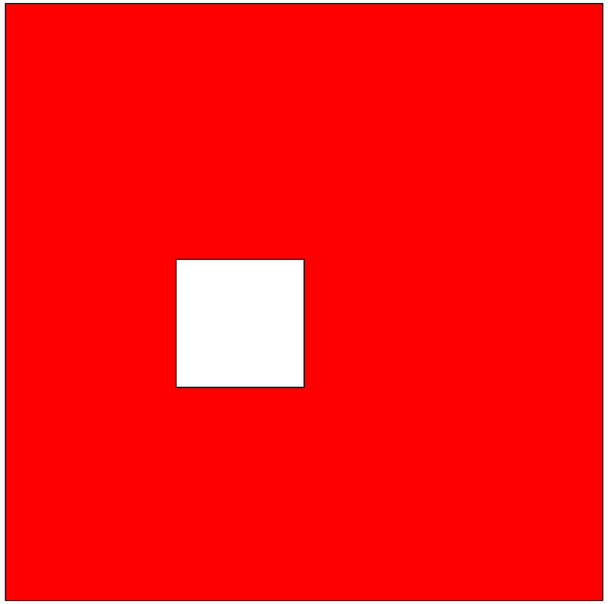

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Algorithm: Single Representative

- Construct WSPD $P_{S,8}$
- For each pair, $P = (X, Y) \in P_{S,8}$
  - Place a set of balls with radius $2^i \ell$ for $-2 \leq i \leq \lceil \log(1/\epsilon) + 1 \rceil$
  - For each ball b take all quadtree boxes which intersect it and are smaller than $r_b \epsilon/(16d)$
  - Store in BBD along with a representative point

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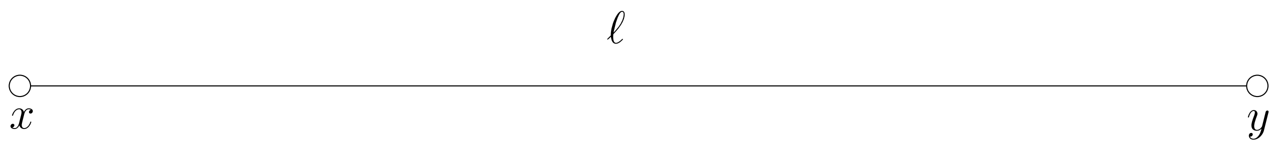

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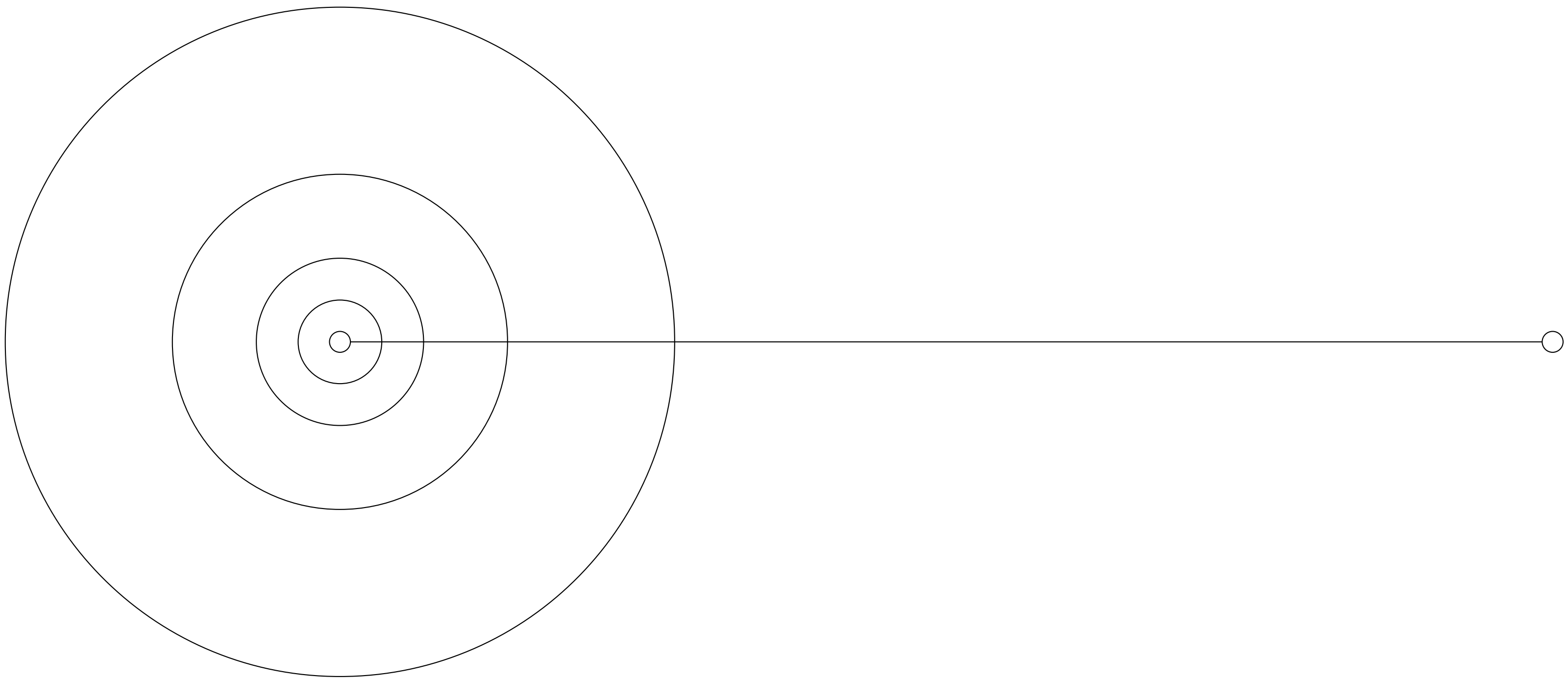

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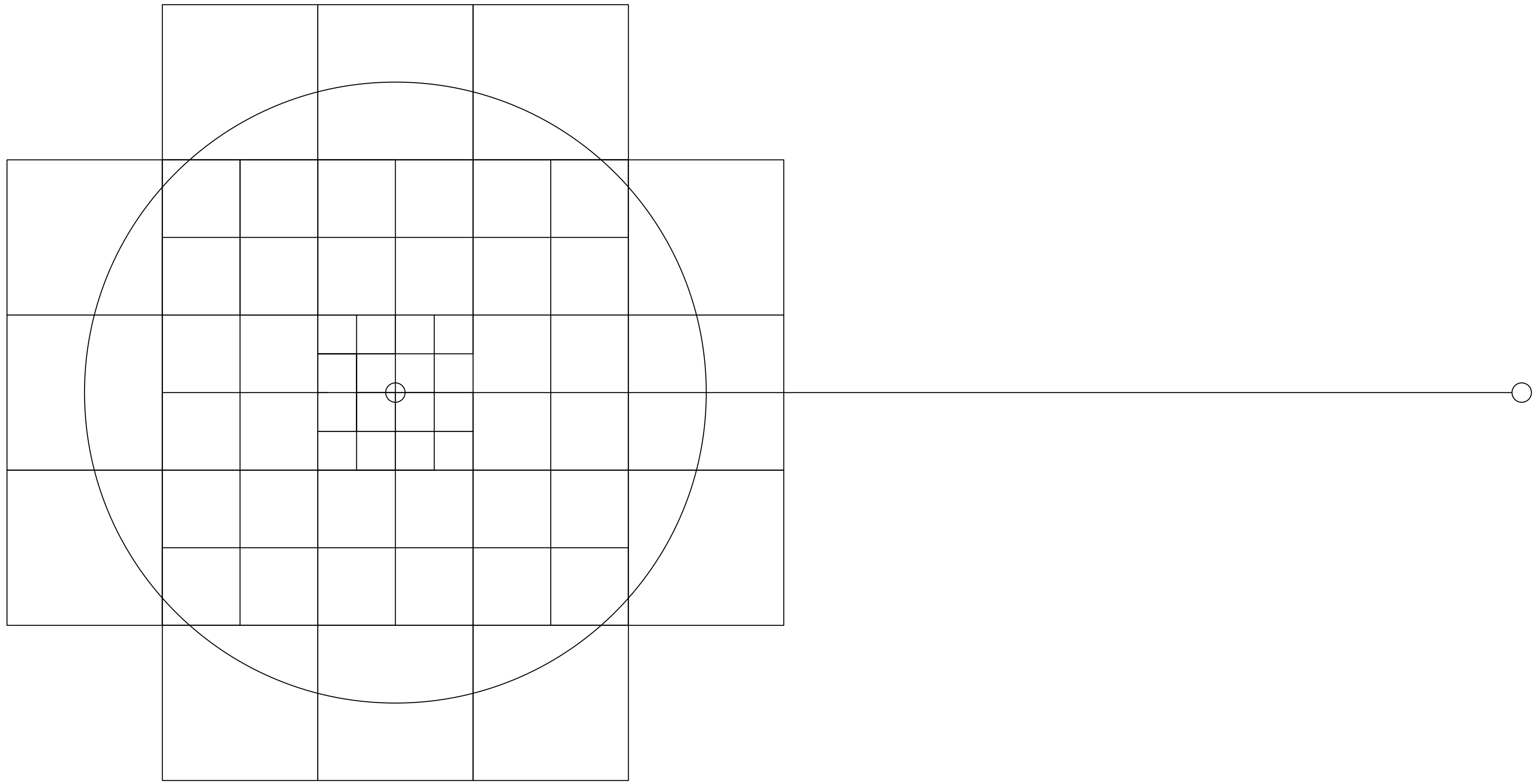
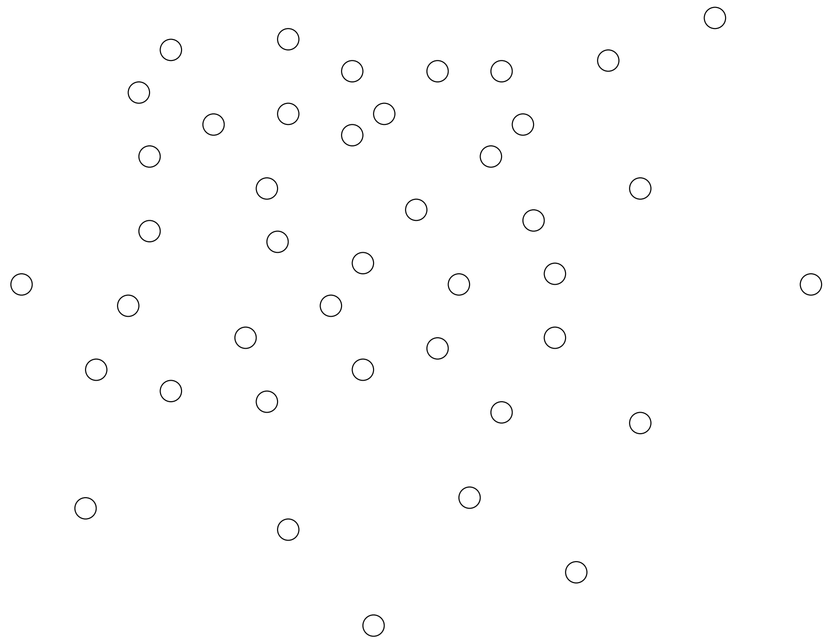


Outside Case



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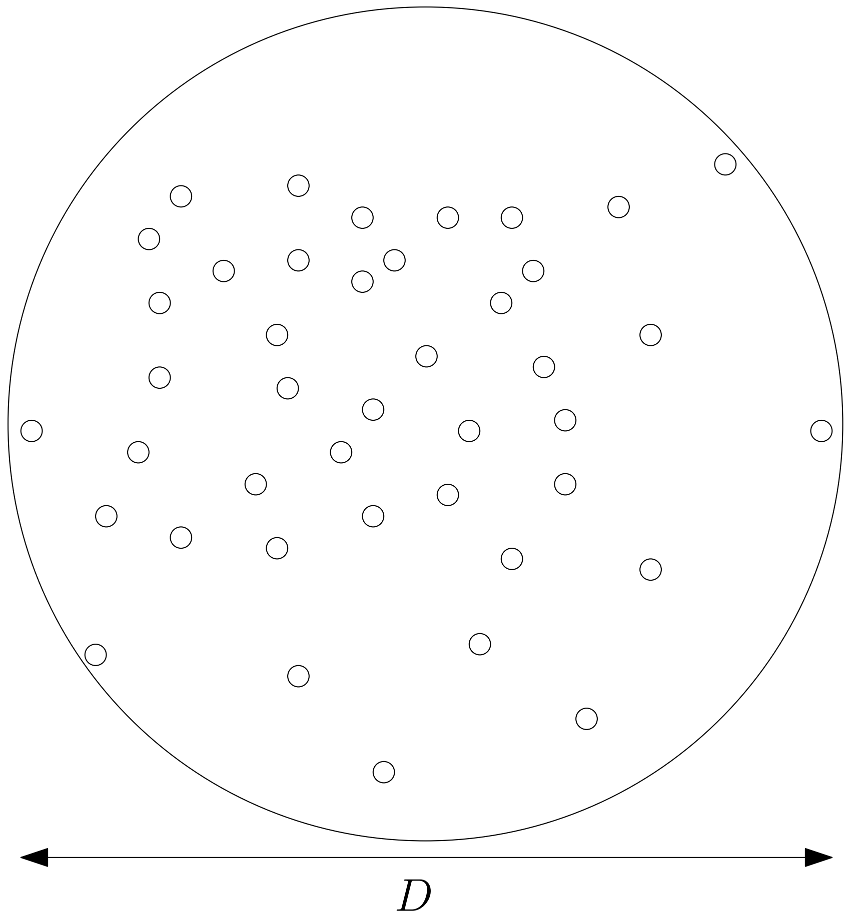

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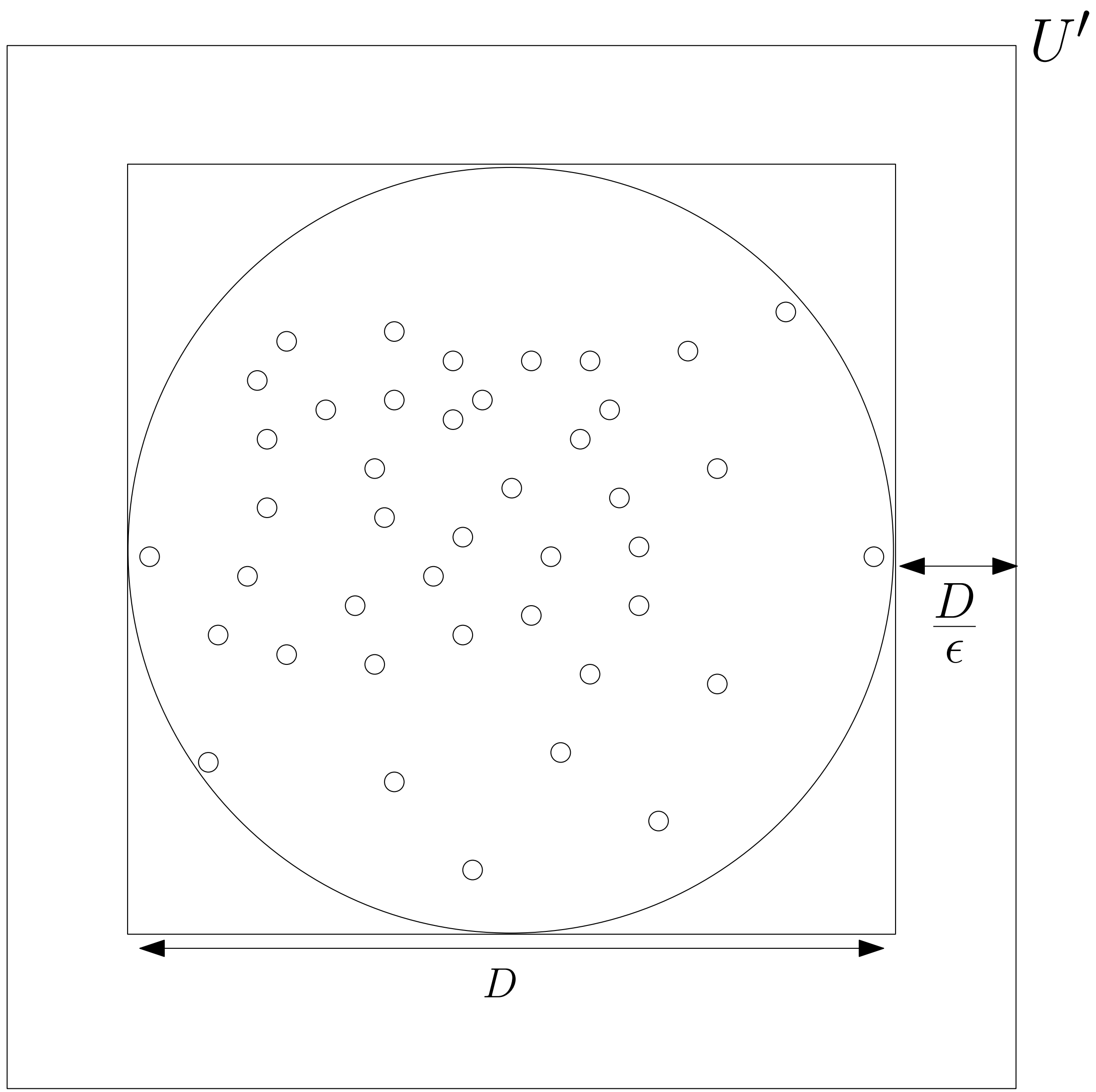

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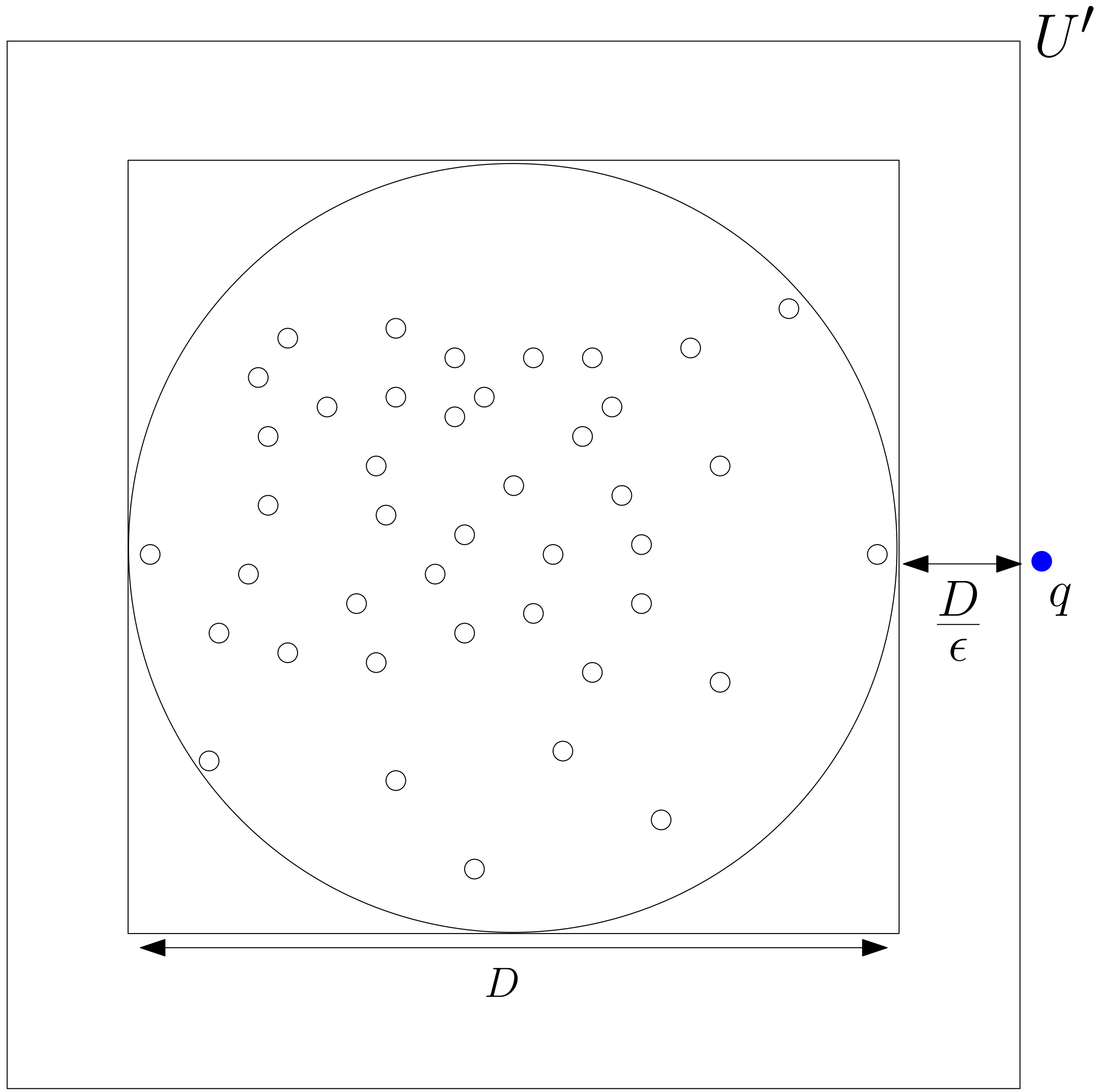

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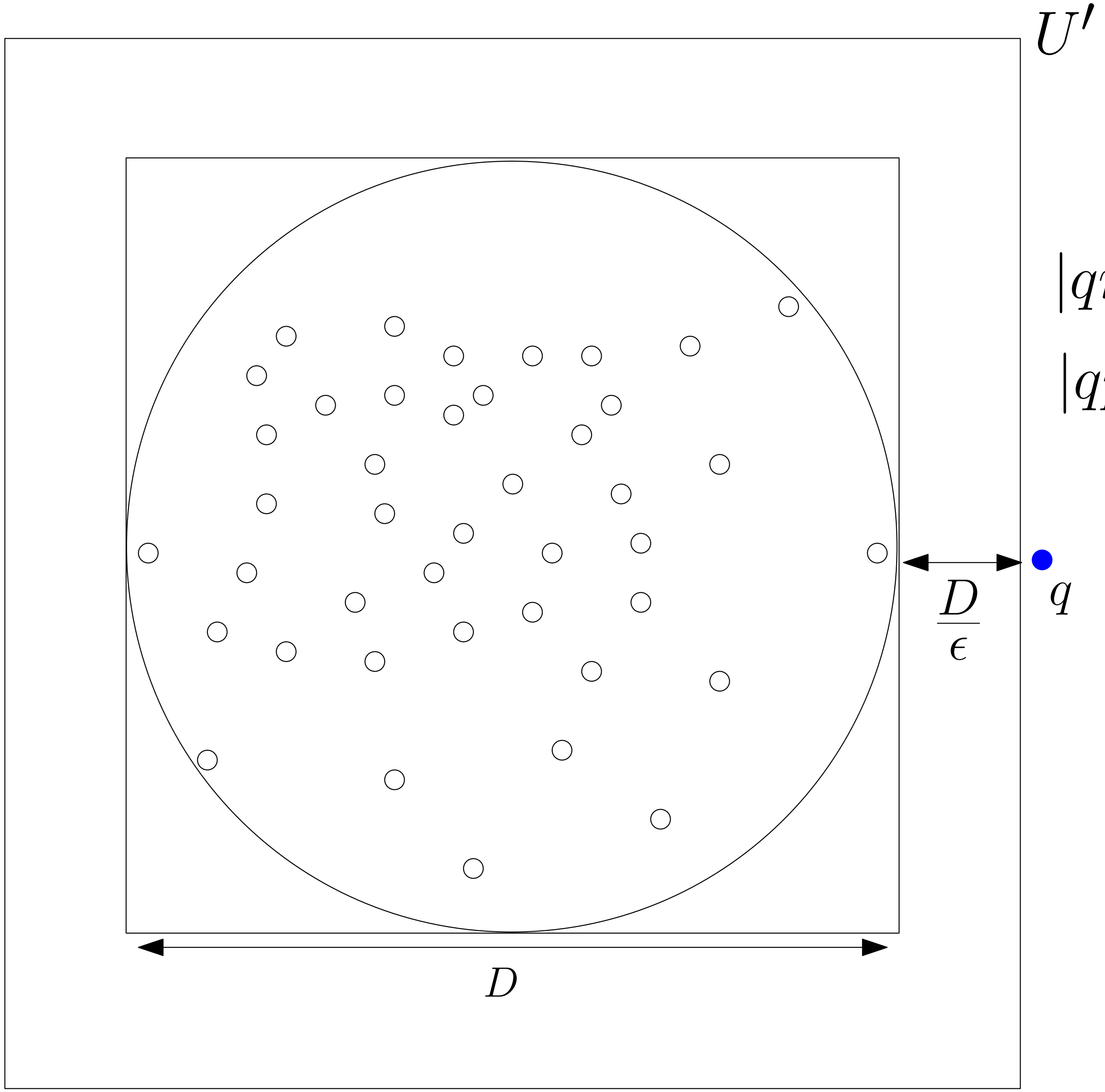

$$|qn| \geq \frac{D}{\epsilon}$$

$$|qp| \leq D + \frac{D}{\epsilon} =$$

$$= \epsilon|qn| + |qn| =$$

$$= (1+\epsilon)|qn|$$

Any $\epsilon/4$ NN is good

Lemma 3.1. Let S be a set of n points in R^d and let $0 < \epsilon \leq 1/2$ be a real parameter. Let x_1 be a point inside a d-cube c of size $(\epsilon/(4d))|x_1y_1|$, where y_1 denotes the nearest neighbor of x_1. If y_2 is an $(\epsilon/4)$-NN of some point x_2 inside c, then y_2 is an ϵ-NN of x_1.

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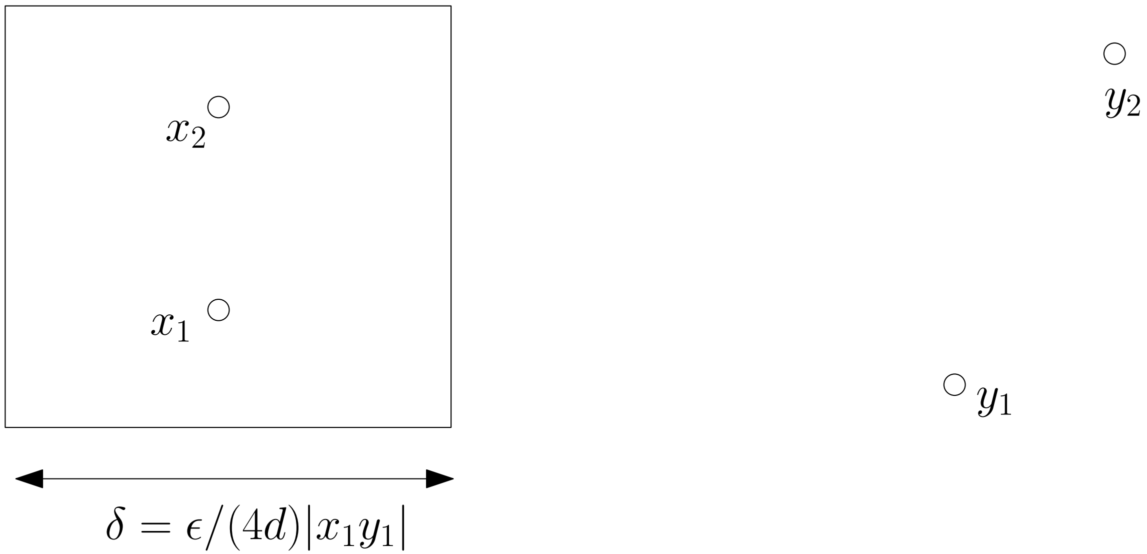

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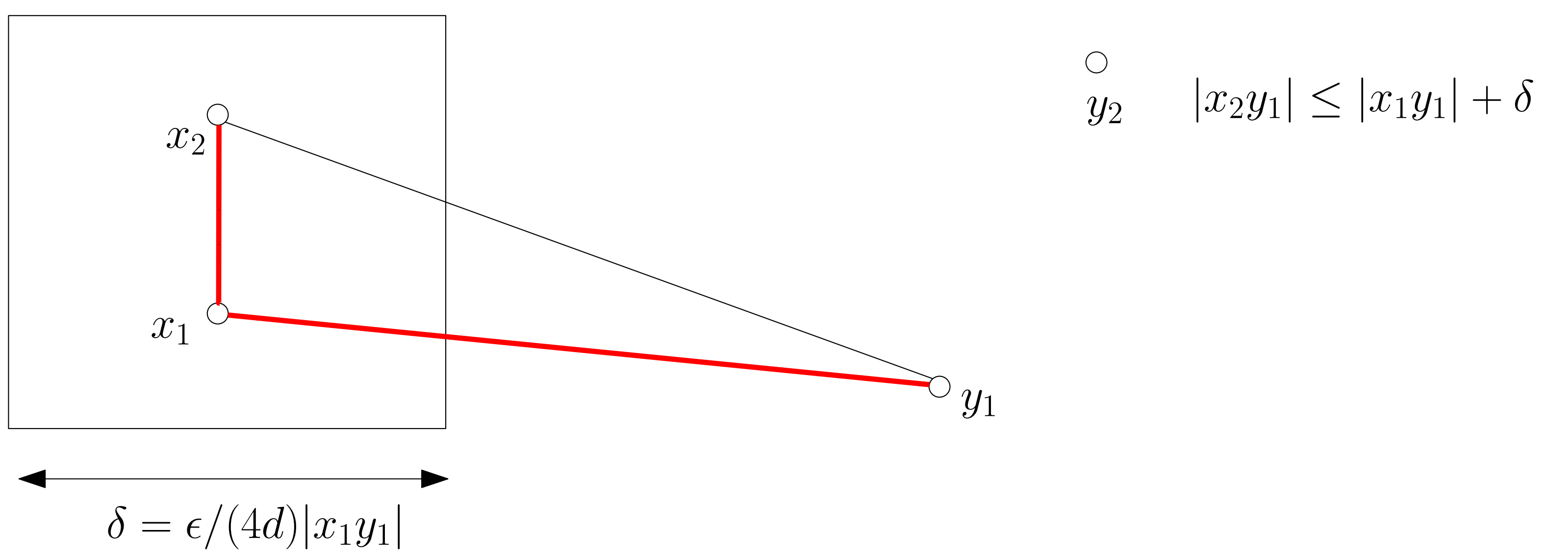

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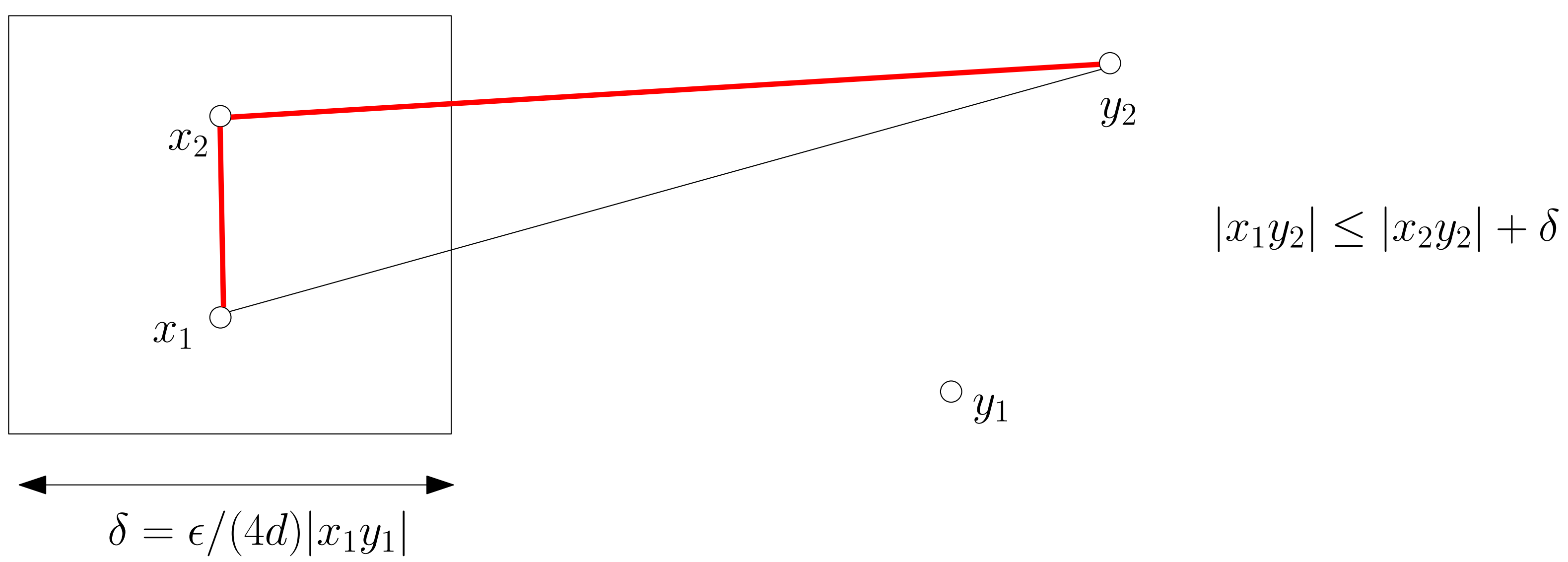

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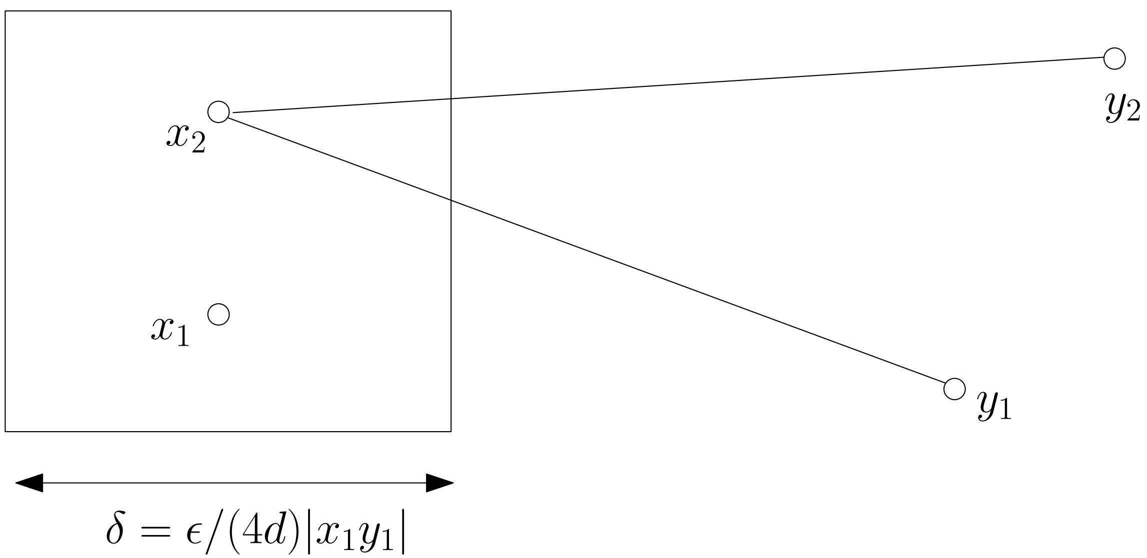

$|x_2 y_2| \leq (1 + \epsilon/4)|x_2 y_1|$

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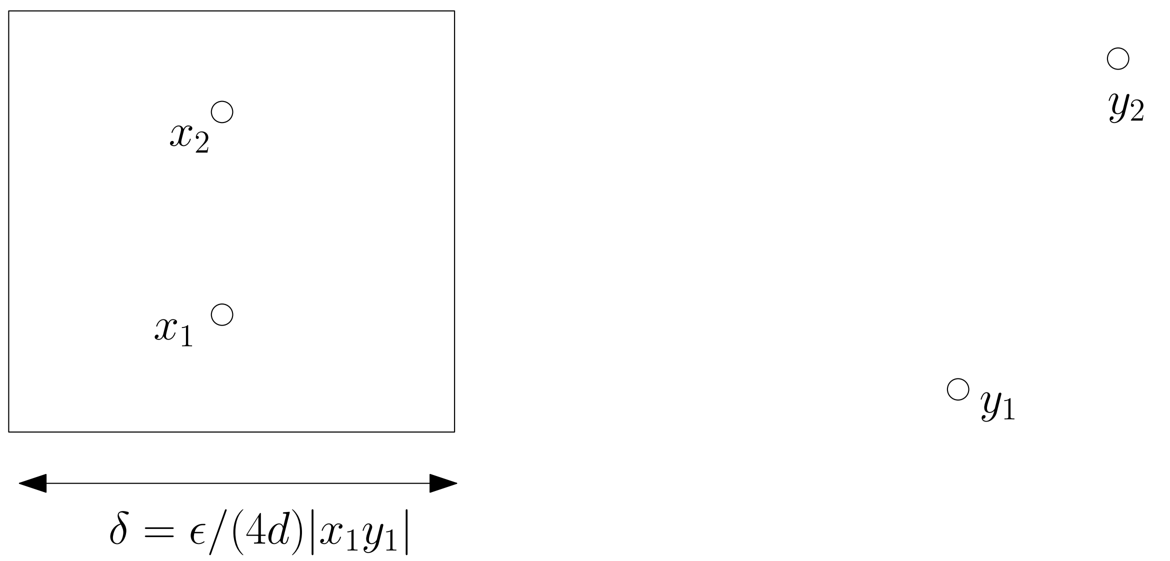

$$|x_2y_1| \leq |x_1y_1| + \delta$$

$$|x_1y_2| \leq |x_2y_2| + \delta$$

$$|x_2y_2| \leq (1+\epsilon/4)|x_2y_1|$$

$$|x_1y_2| \leq (1+\epsilon/4)(1+\epsilon/4)|x_1y_1| \leq (1+\epsilon)|x_1y_1|$$

$(1, \epsilon)$-Approximate Voronoi Diagram

- $P = (X, Y)$
- Query point q
- Notation

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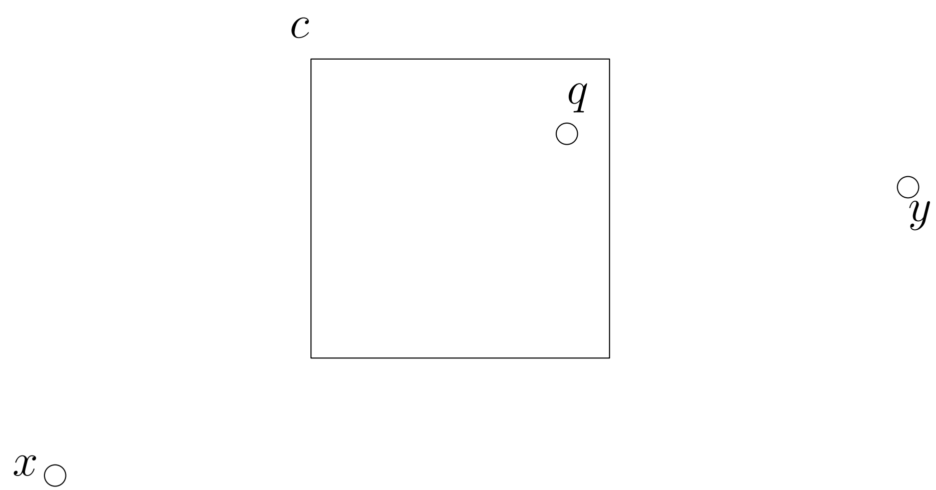

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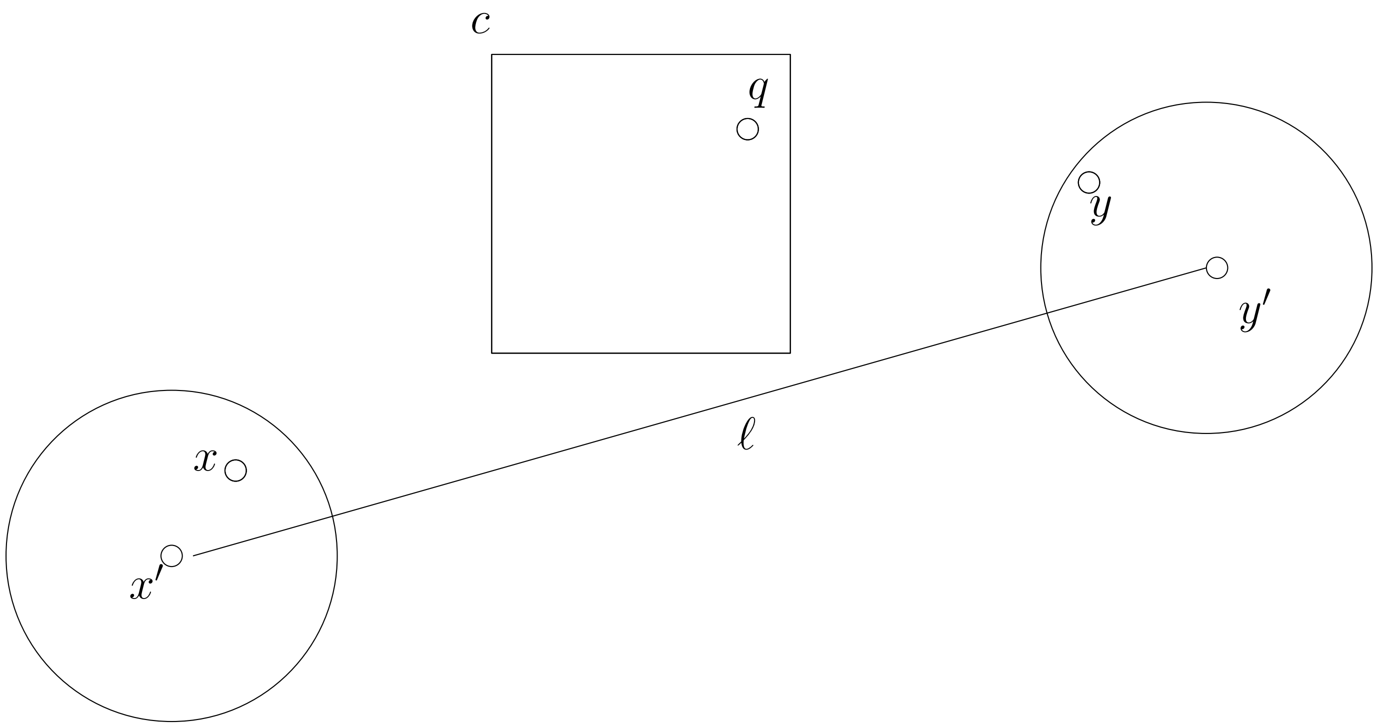
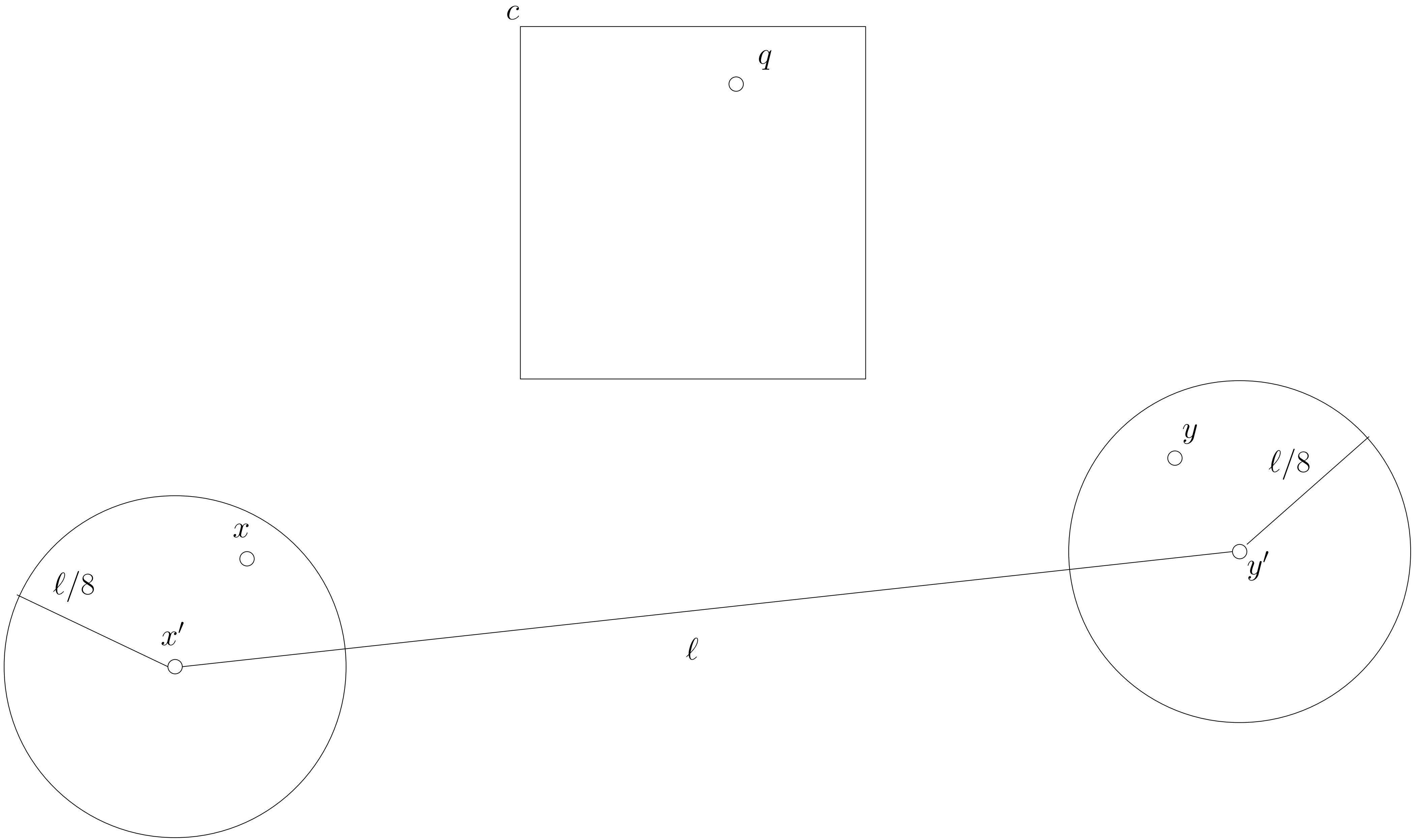
Specifically, look at pair in WSPD which separates x and y



Case 1: $|qy'| \geq 2\ell/\epsilon$



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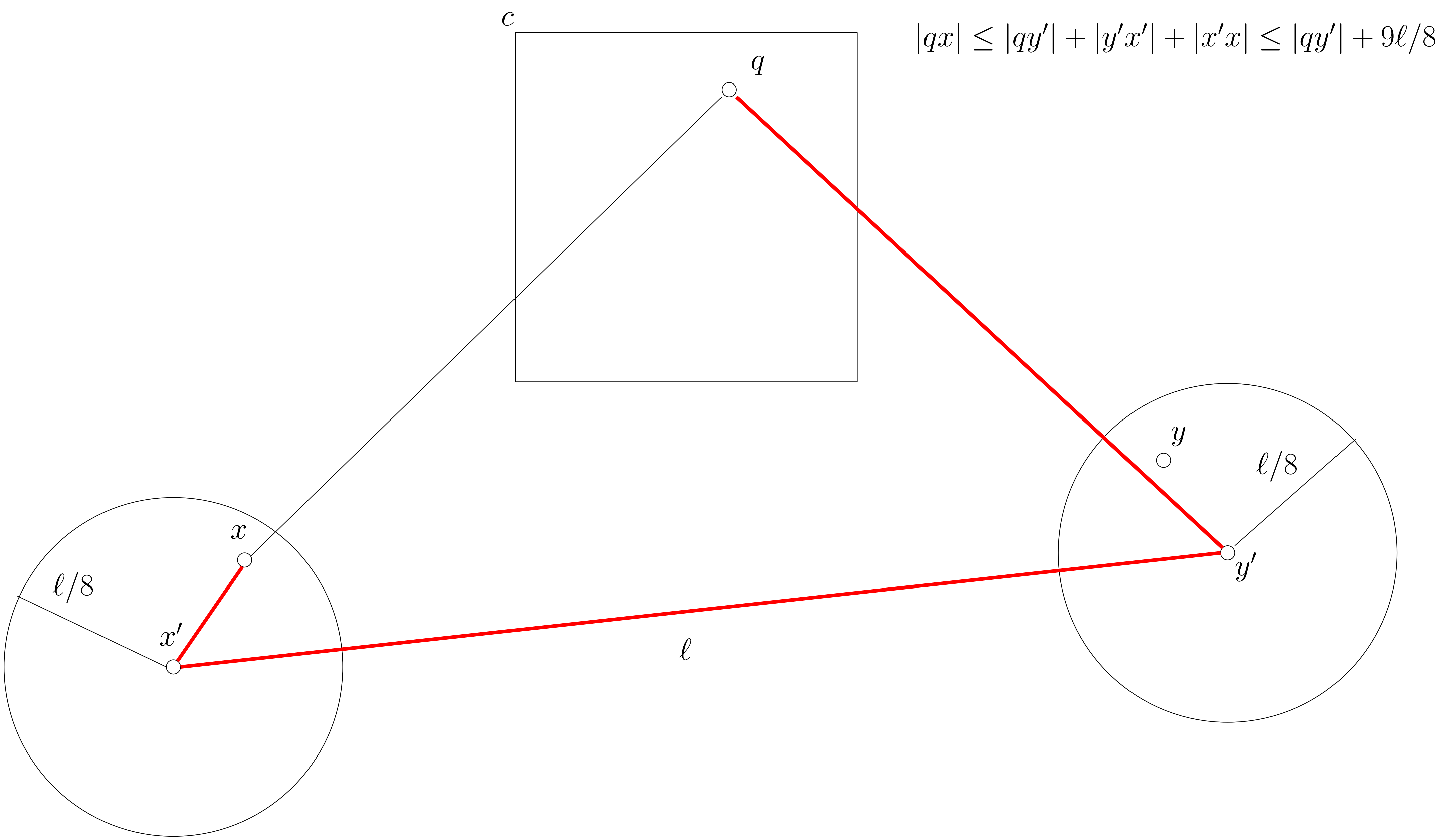

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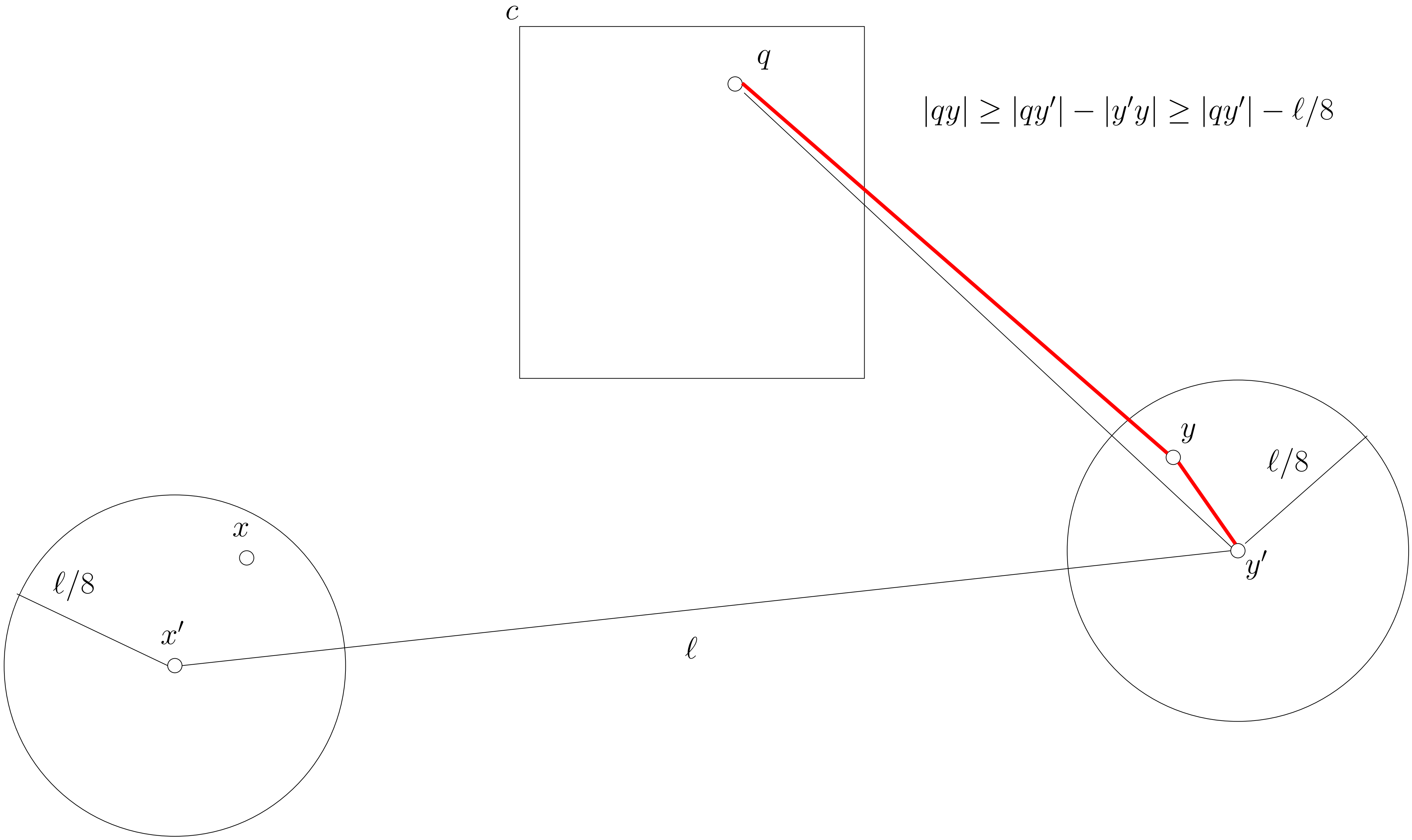

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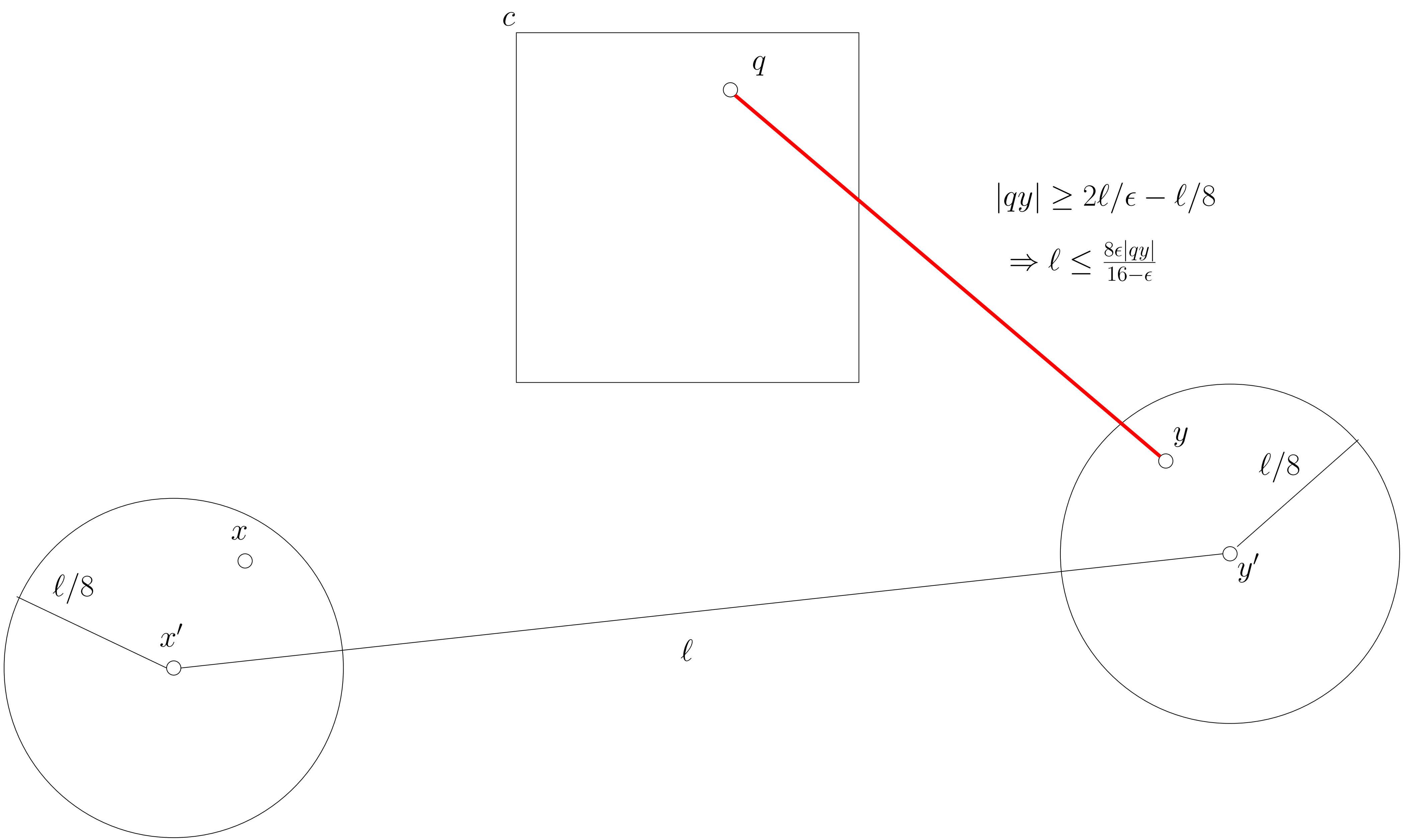

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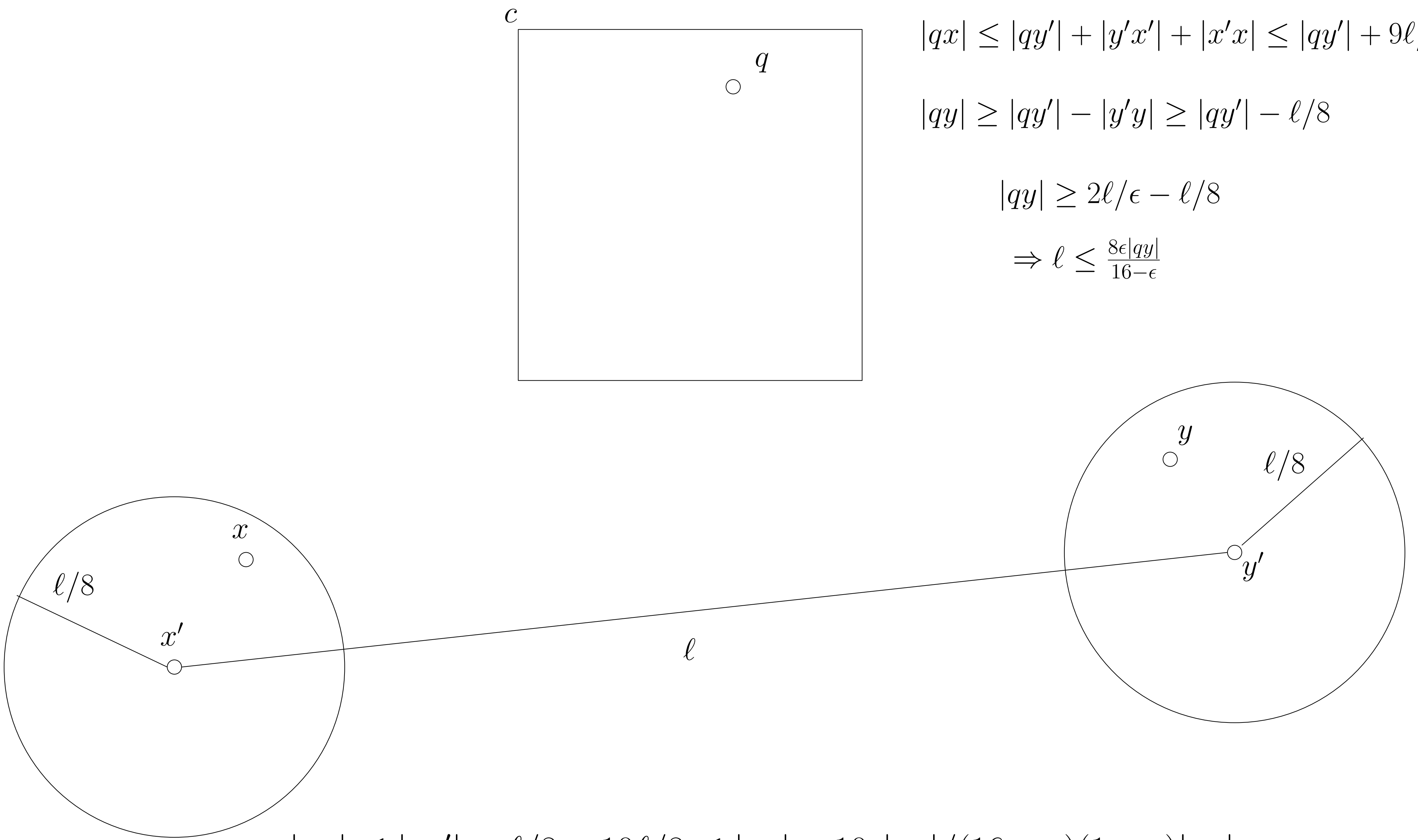
$$|qx| \leq |qy'| + |y'x'| + |x'x| \leq |qy'| + 9\ell/8$$

$$|qy| \geq |qy'| - |y'y| \geq |qy'| - \ell/8$$

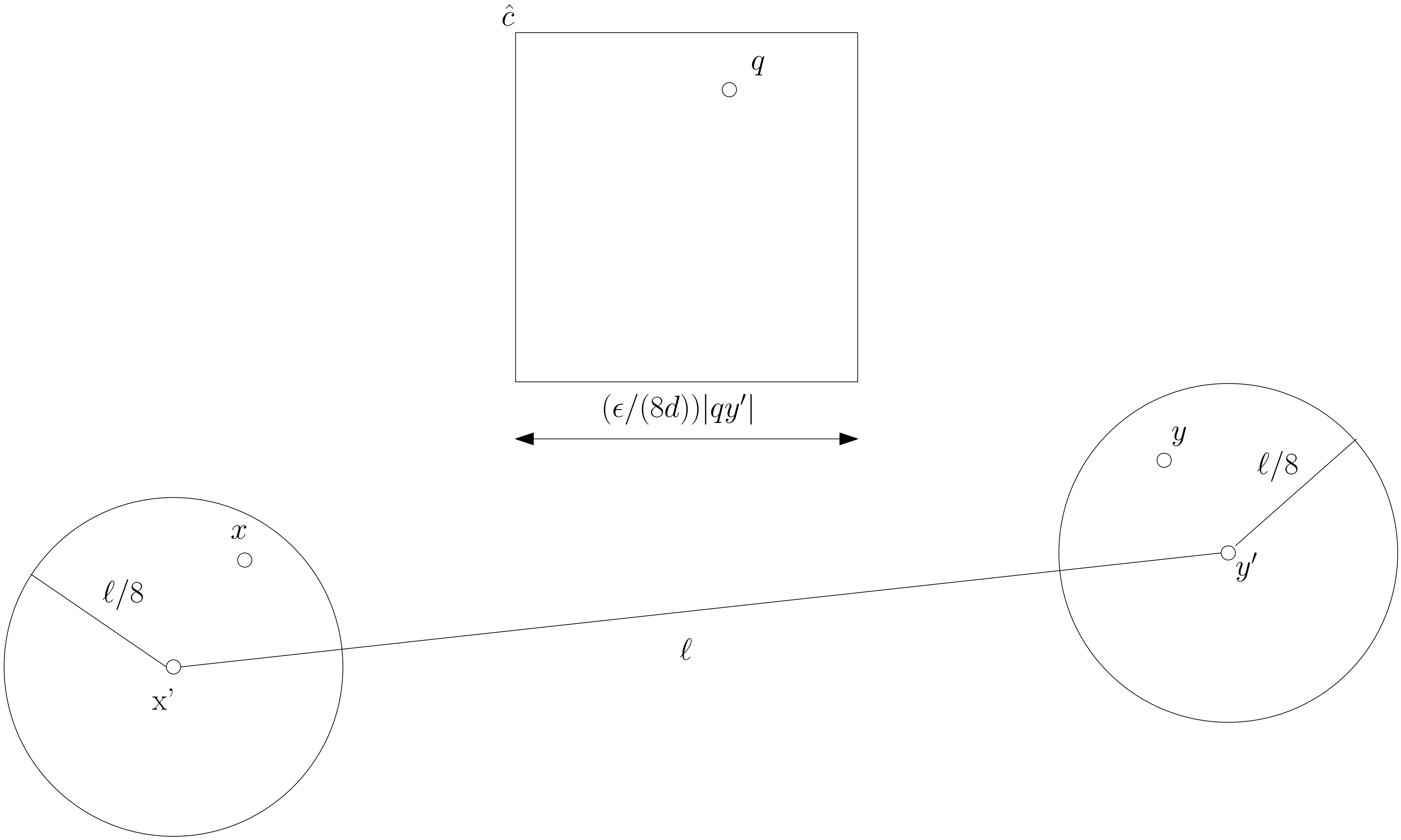
$$|qy| \geq 2\ell/\epsilon - \ell/8$$

$$\Rightarrow \ell \leq \frac{8\epsilon|qy|}{16-\epsilon}$$



$$|qx| \leq |qy'| - \ell/8 + 10\ell/8 \leq |qy| + 10\epsilon|qy|/(16-\epsilon)(1+\epsilon)|qy|$$

Case 2: $\ell \leq |qy'| < 2\ell/\epsilon$



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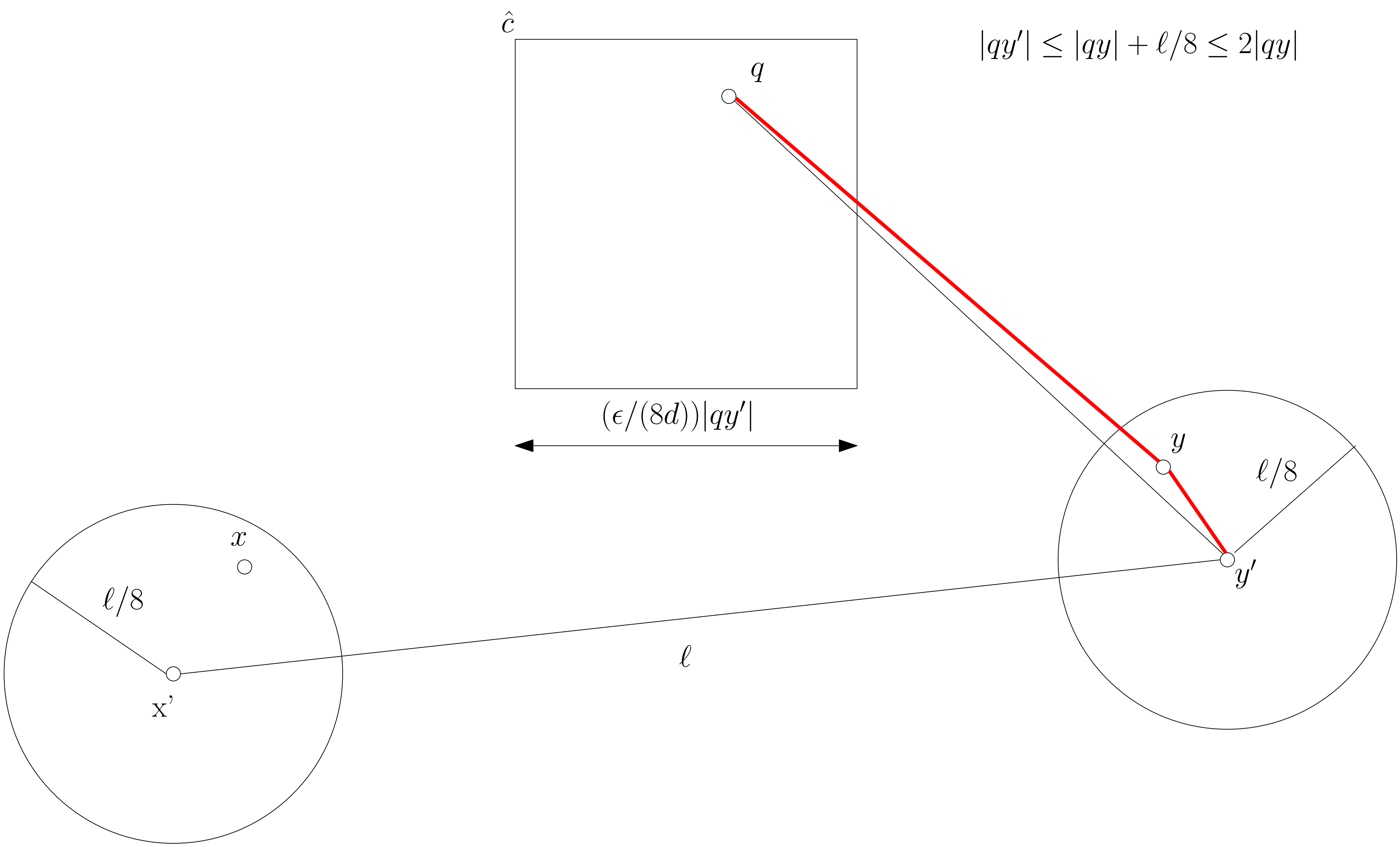

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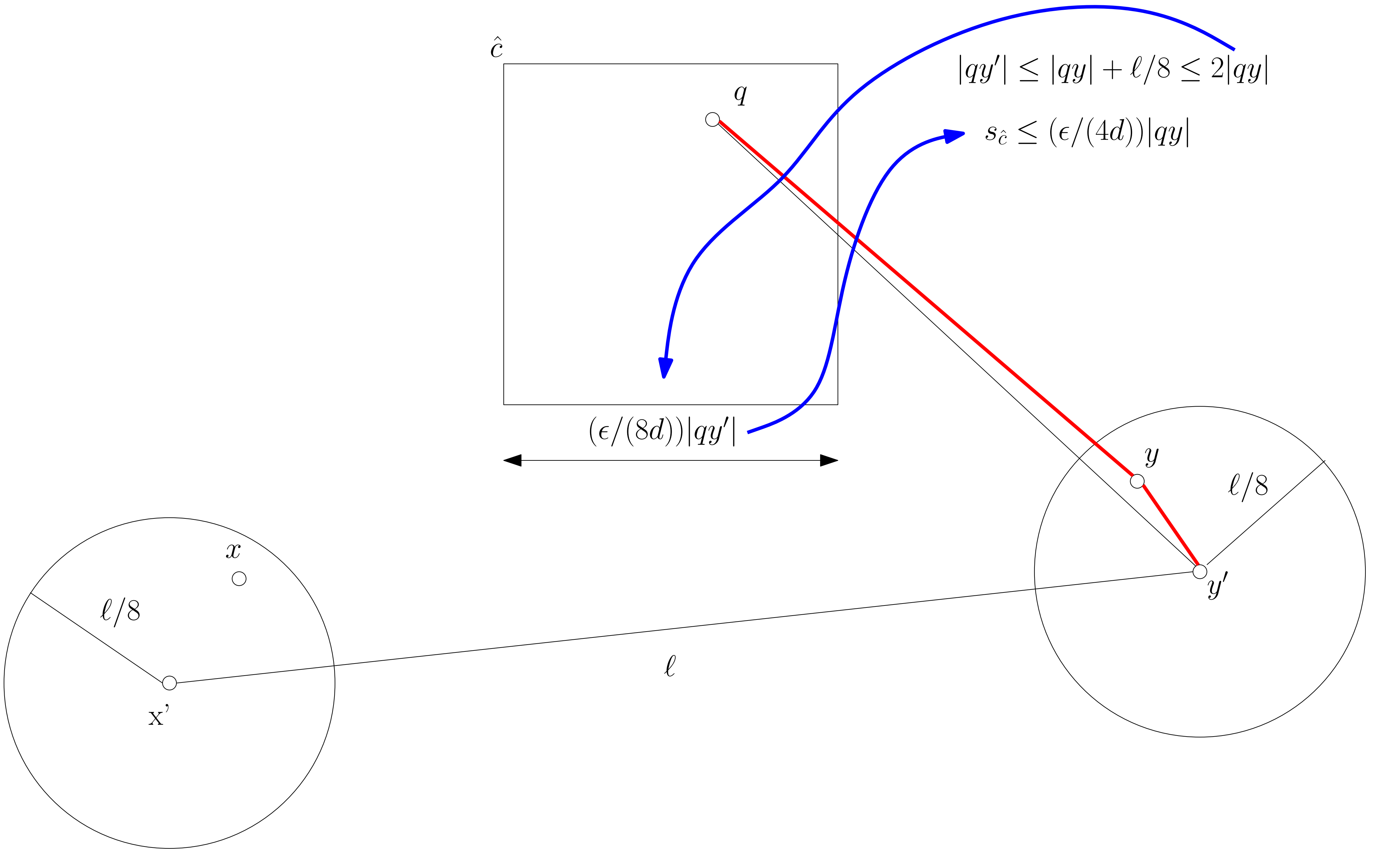

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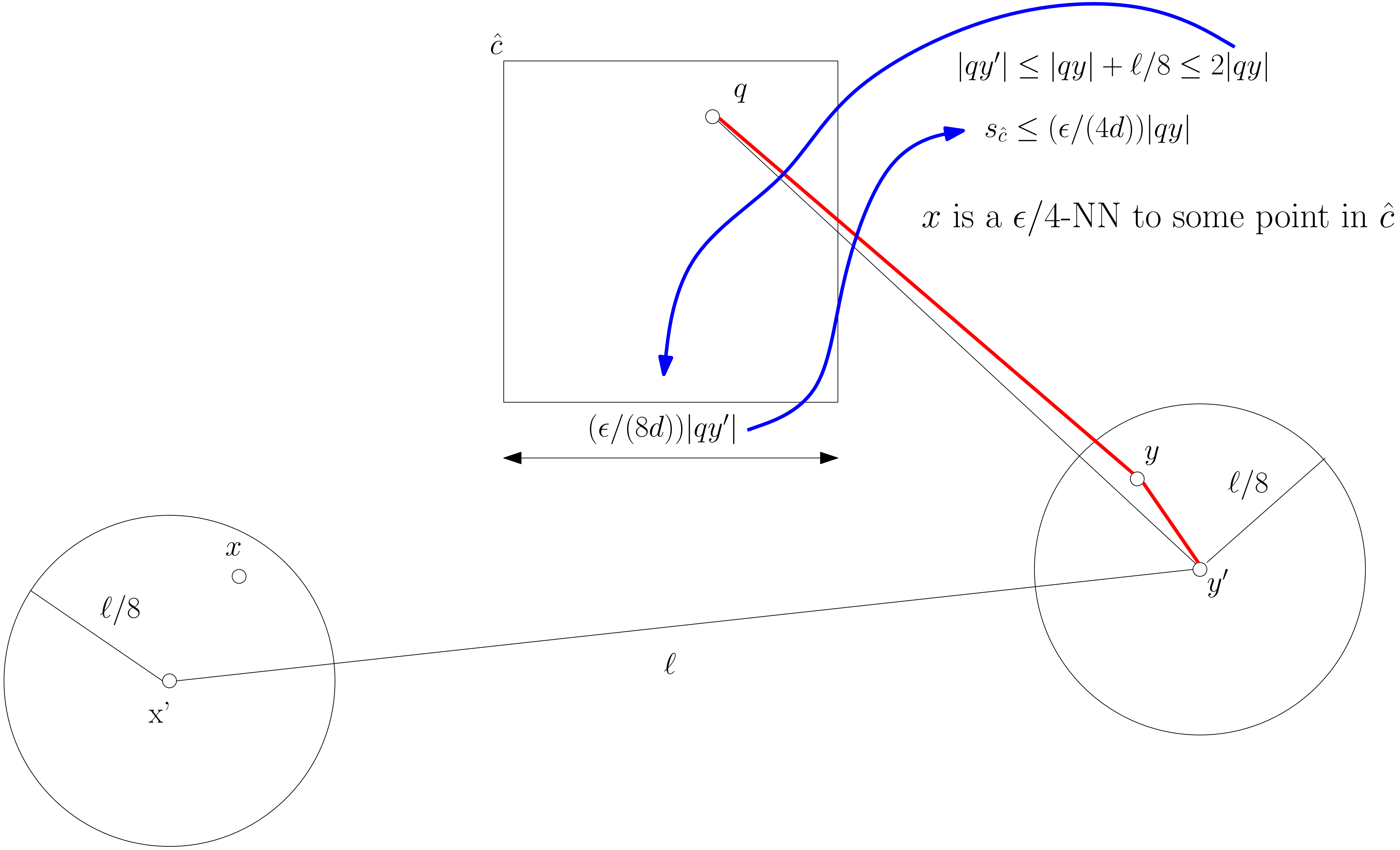
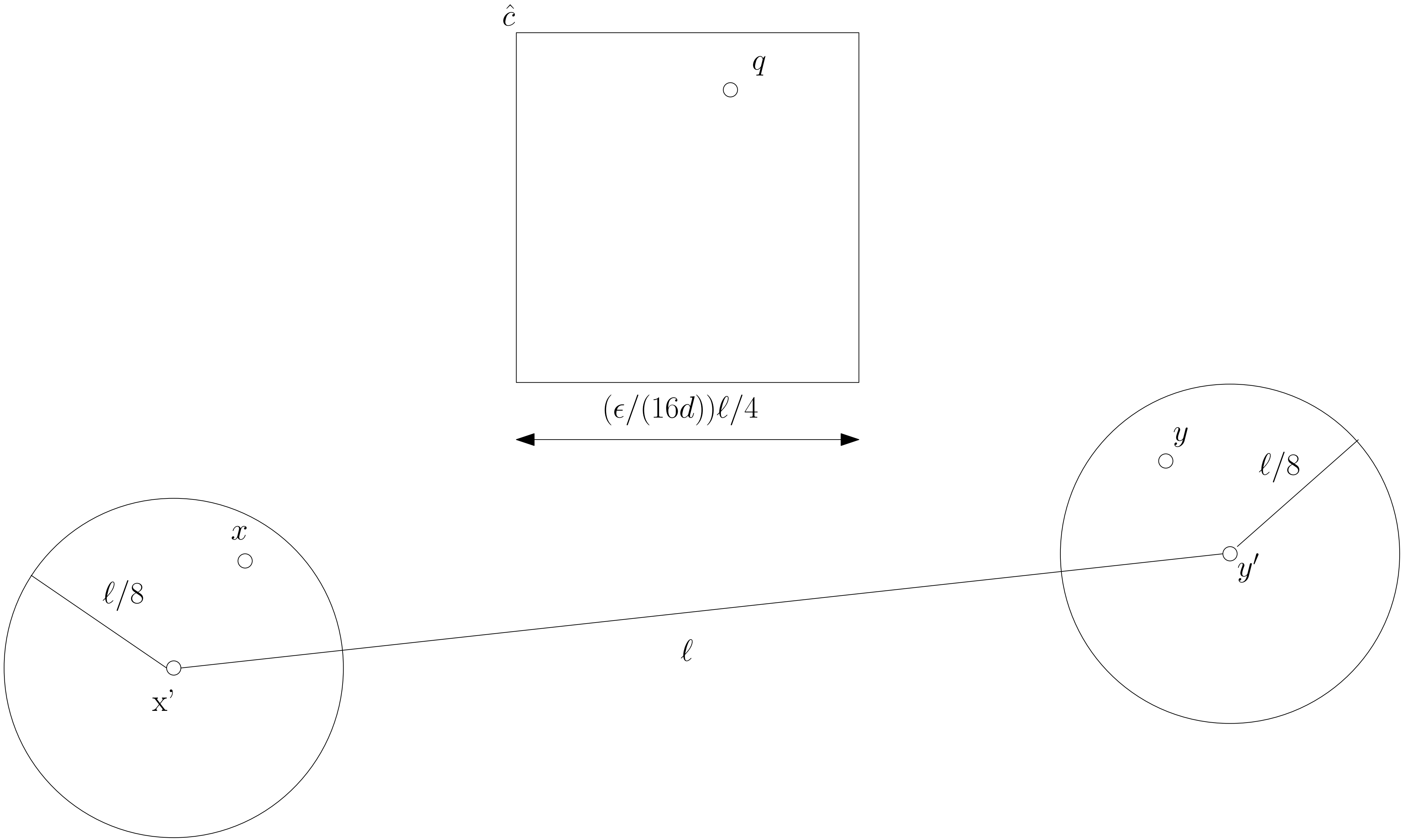


Case 3: $|qy'| < \ell/4$



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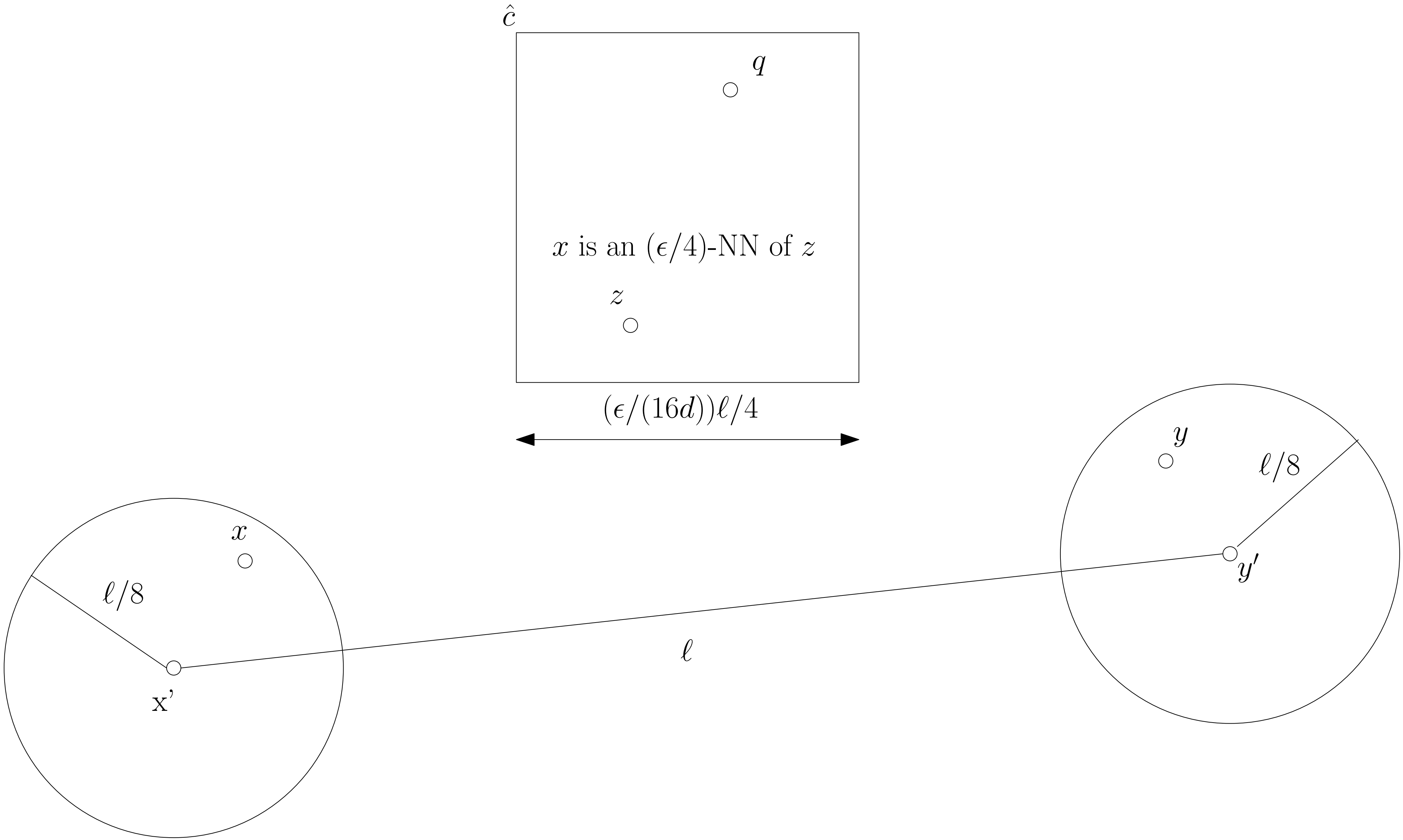

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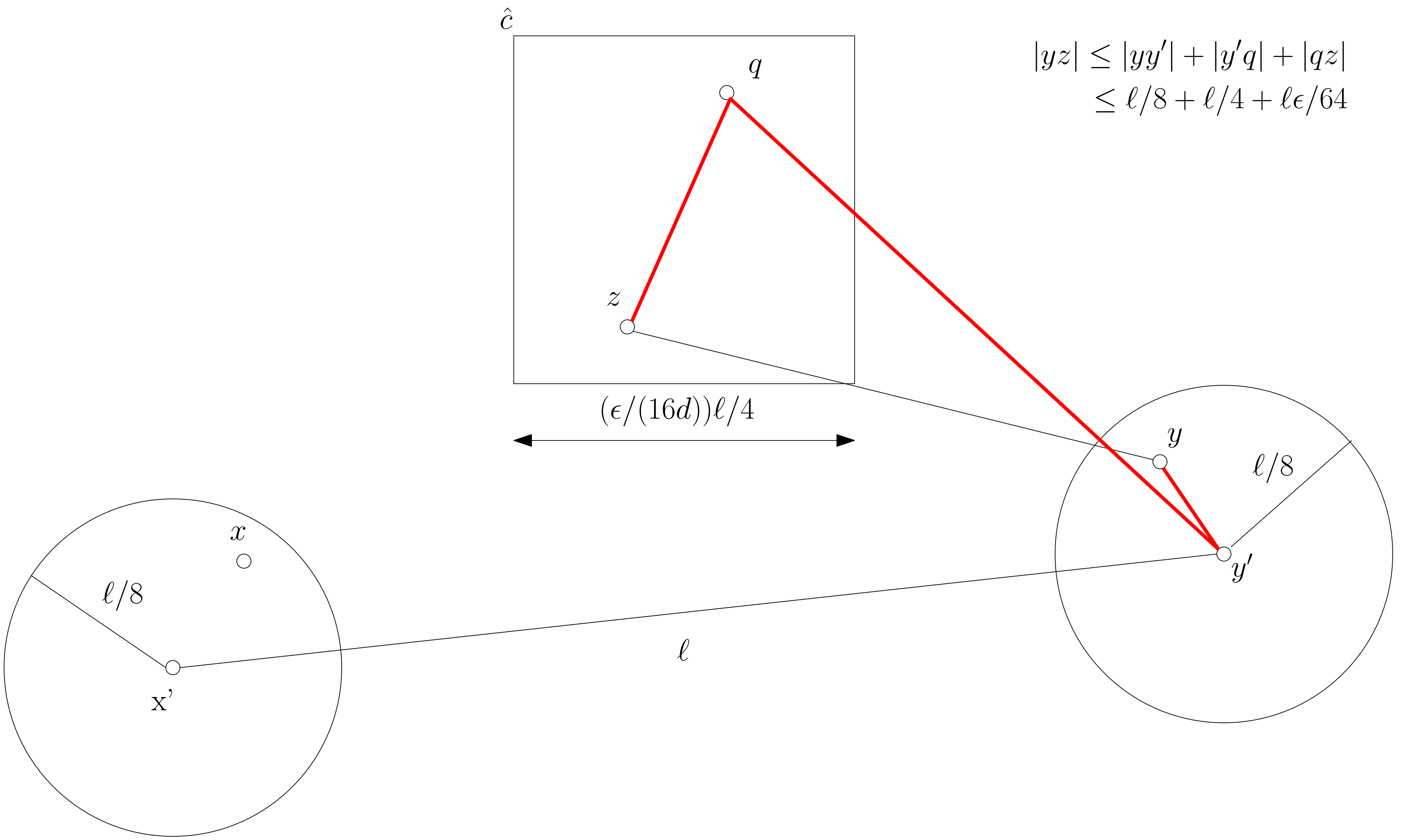

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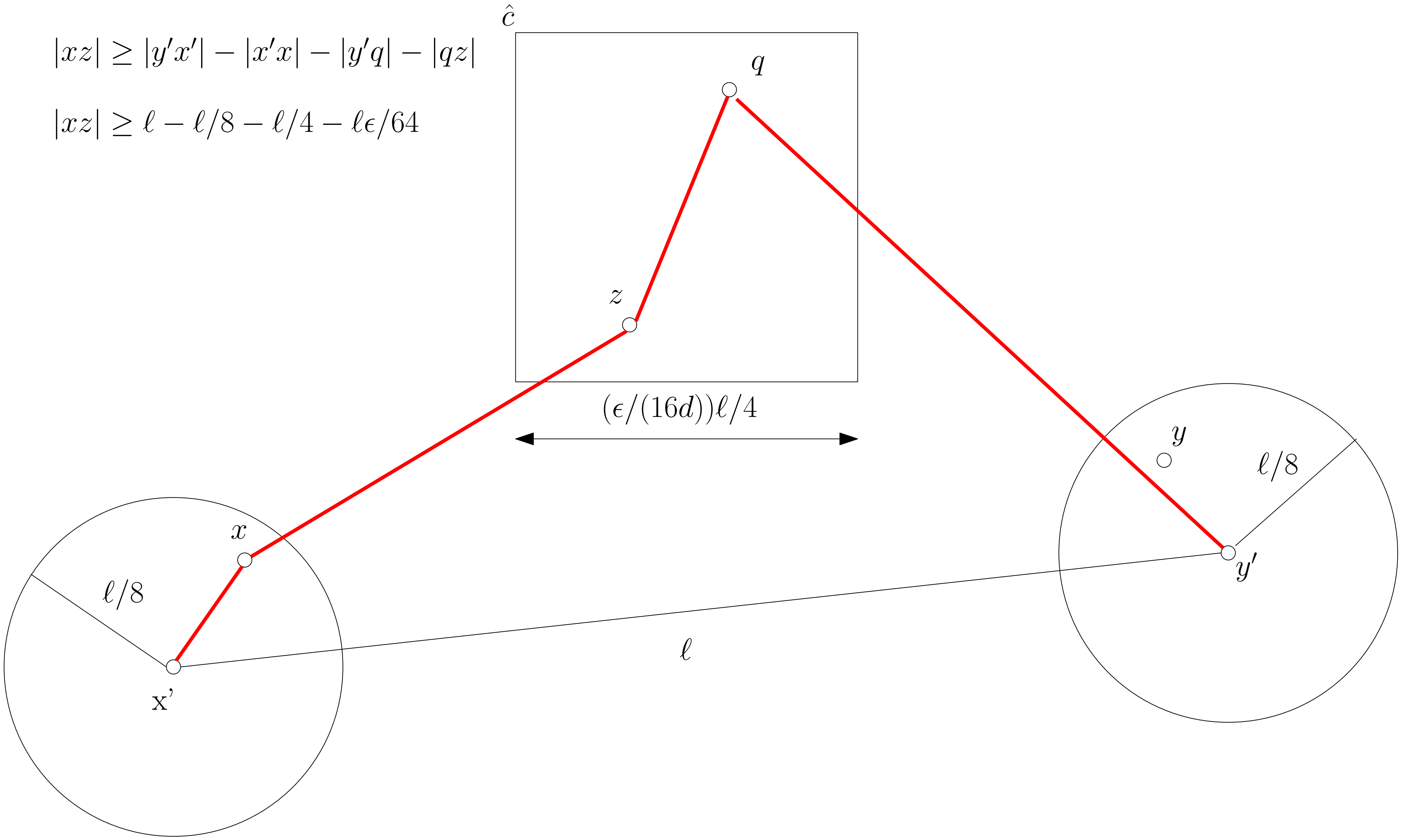
$$|xz| \geq |y'x'| - |x'x| - |y'q| - |qz|$$

$$|xz| \geq \ell - \ell/8 - \ell/4 - \ell\epsilon/64$$



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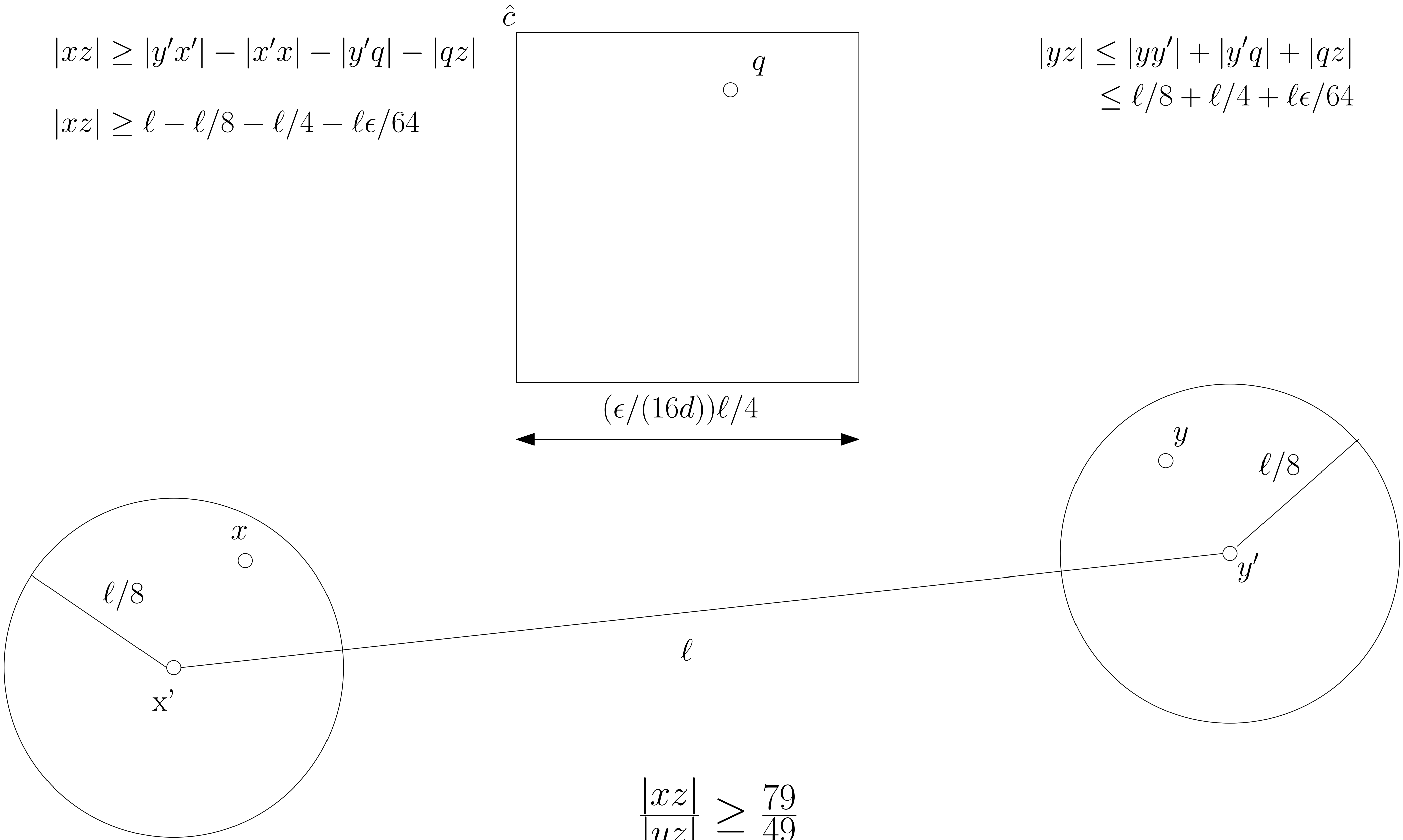
$$|xz| \geq \ell - \ell/8 - \ell/4 - \ell\epsilon/64$$

$$|yz| \leq |yy'| + |y'q| + |qz| \leq \ell/8 + \ell/4 + \ell\epsilon/64$$



$$\frac{|xz|}{|yz|} \geq \frac{79}{49}$$

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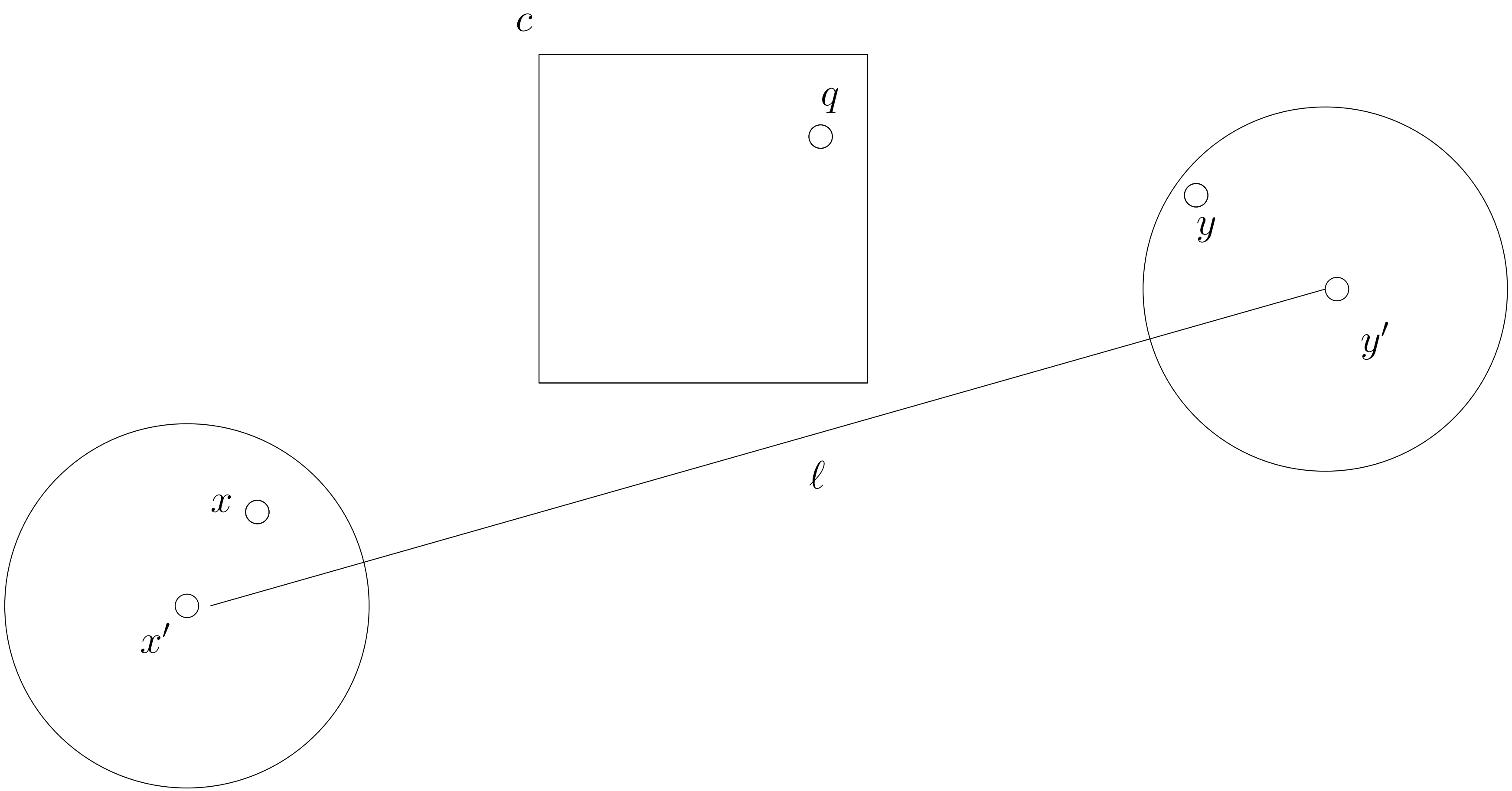


Choosing representatives: x is a $\epsilon/4$-NN to any point in c

If we return x, either it is the nearest neighbor of q or an ϵ-NN of q

Time and Space Bounds

Space Bounds

- $O(n)$ pairs in WSPD
- $O(\frac{1}{\epsilon^d})$ cells per ball
- $O(\log(\frac{1}{\epsilon}))$ balls per pair
- $\Rightarrow O(\frac{n}{\epsilon^d}\log(\frac{1}{\epsilon}))$ cells

Algorithm

- Construct WPSD $P_{S,8}$
- For each pair, $P = (X, Y) \in P_{S,8}$
  - Place a set of balls with radius $2^i\ell$ for $-2 \le i \le \lceil \log(1/\epsilon) + 1 \rceil$
  - For each ball b take all quadtree boxes which intersect it and are smaller than $r_b\epsilon/(16d)$
  - Store in BBD along with a representative point

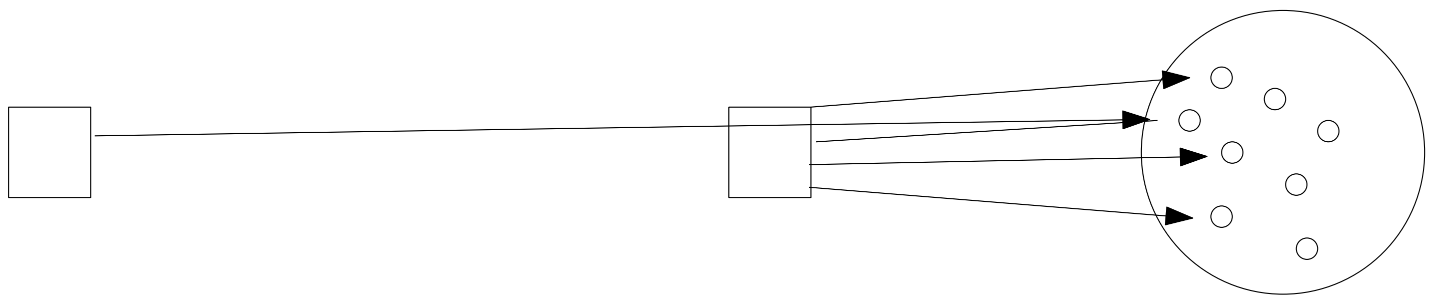
Time Bounds

BBD tree is of depth $\log(\frac{n}{\epsilon^d}\log(\frac{1}{\epsilon})) = \log(n/\epsilon)$

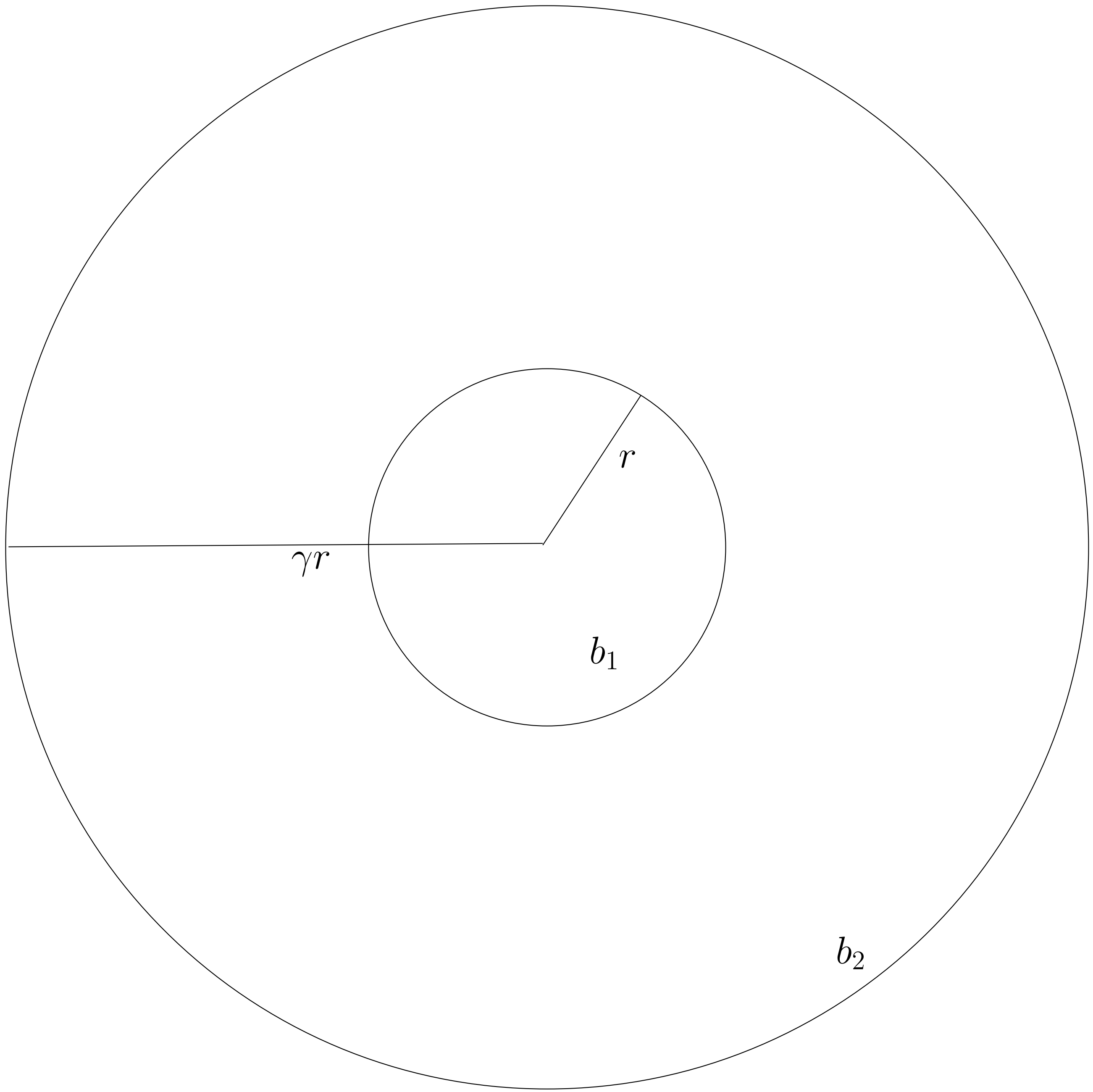
Algorithm: Multiple Representatives

- Construct WSPD $P_{S,4}$
- For each pair, $3 \leq i \leq \lceil \log \beta + 2 \rceil$
- Keep all overlapping cells not bigger than $\Delta_b = r_b/(32\gamma d)$
- Store in BBD tree along with $t > 1$ representatives

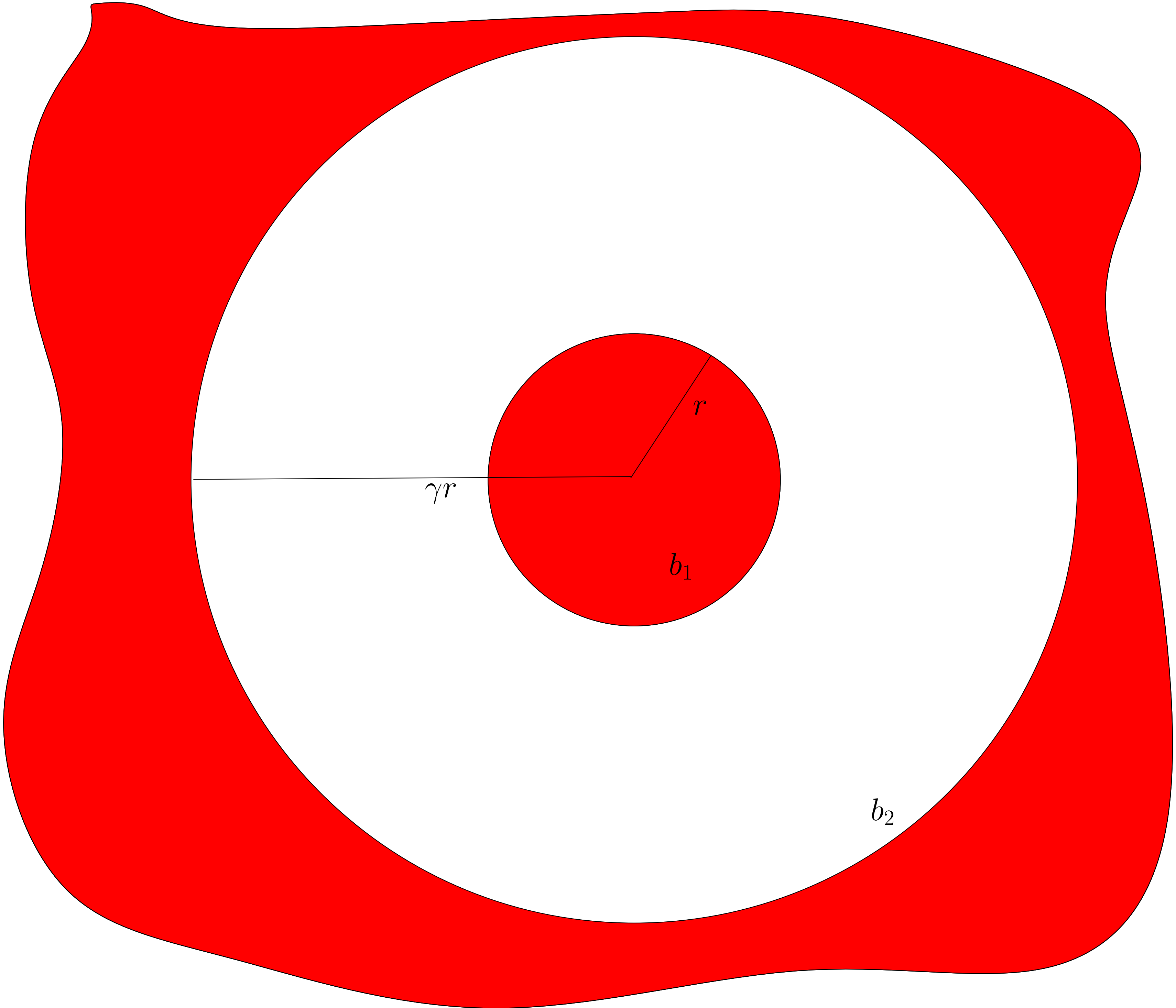
Idea:*If we allow the more representatives, need fewer cells*



Concentric Ball Lemma



Concentric Ball Lemma



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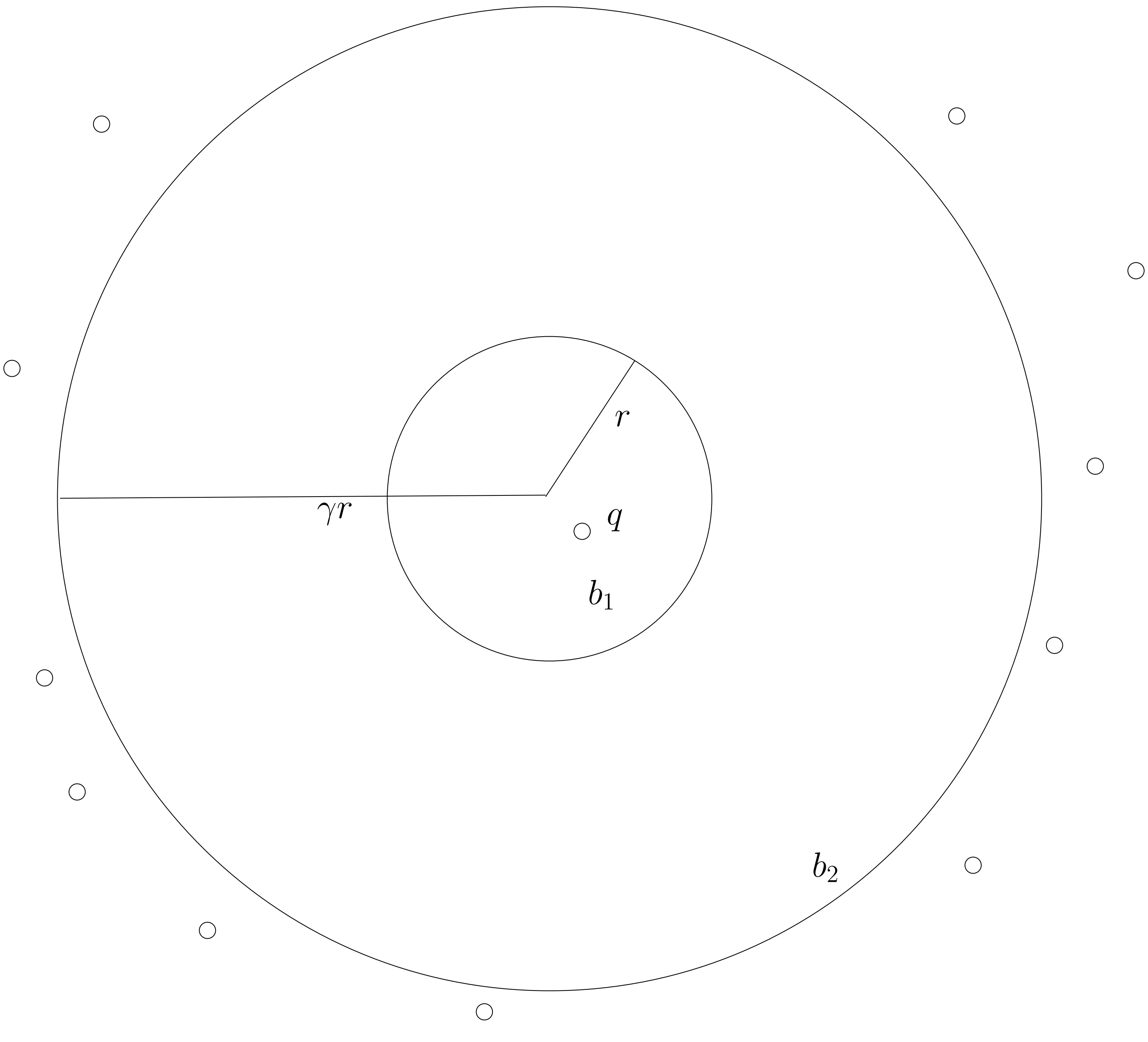
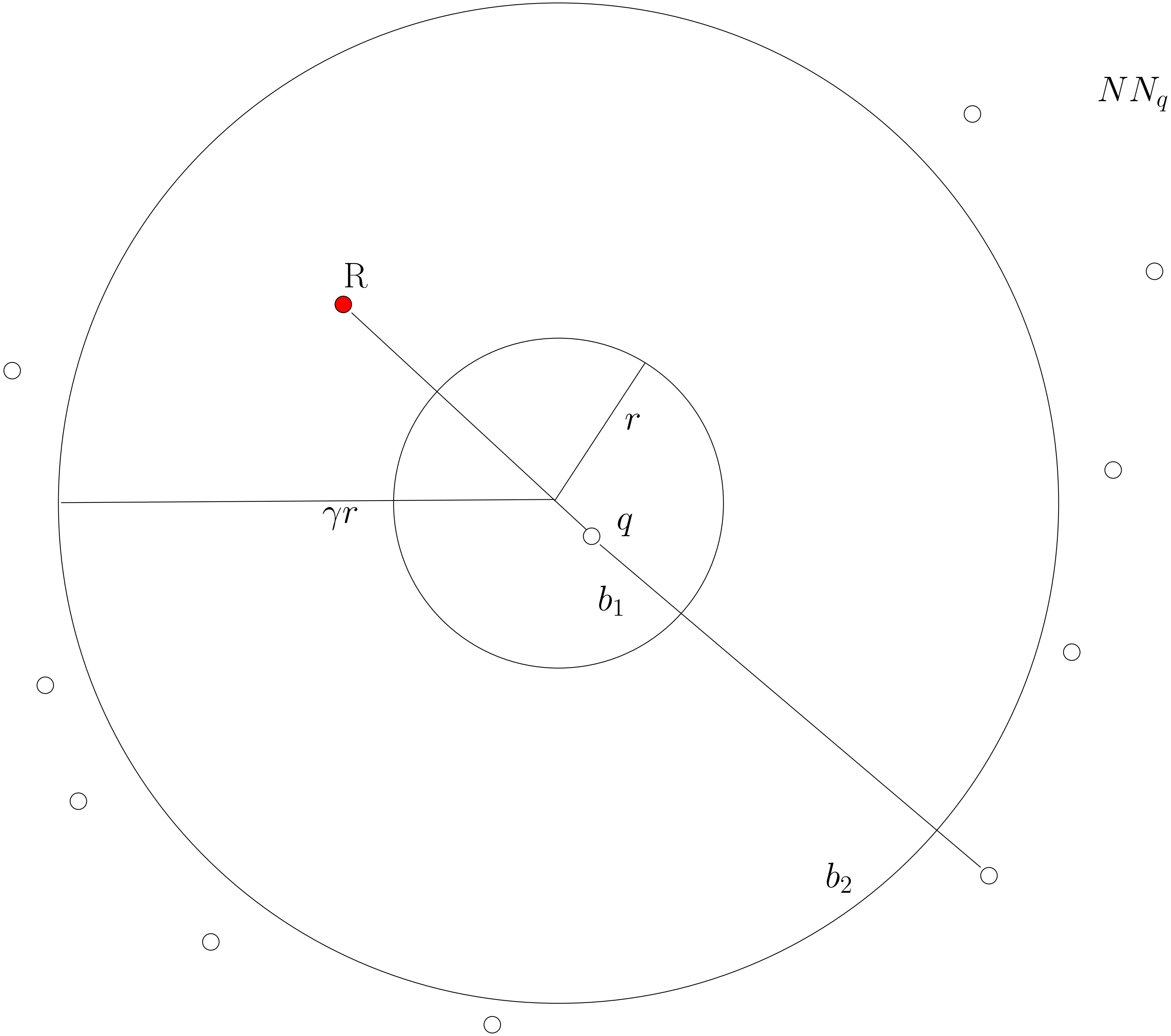


Concentric Ball Lemma

$$NN_q(R) \leq (1+\epsilon)NN_q(S \cap \overline{b_2})$$



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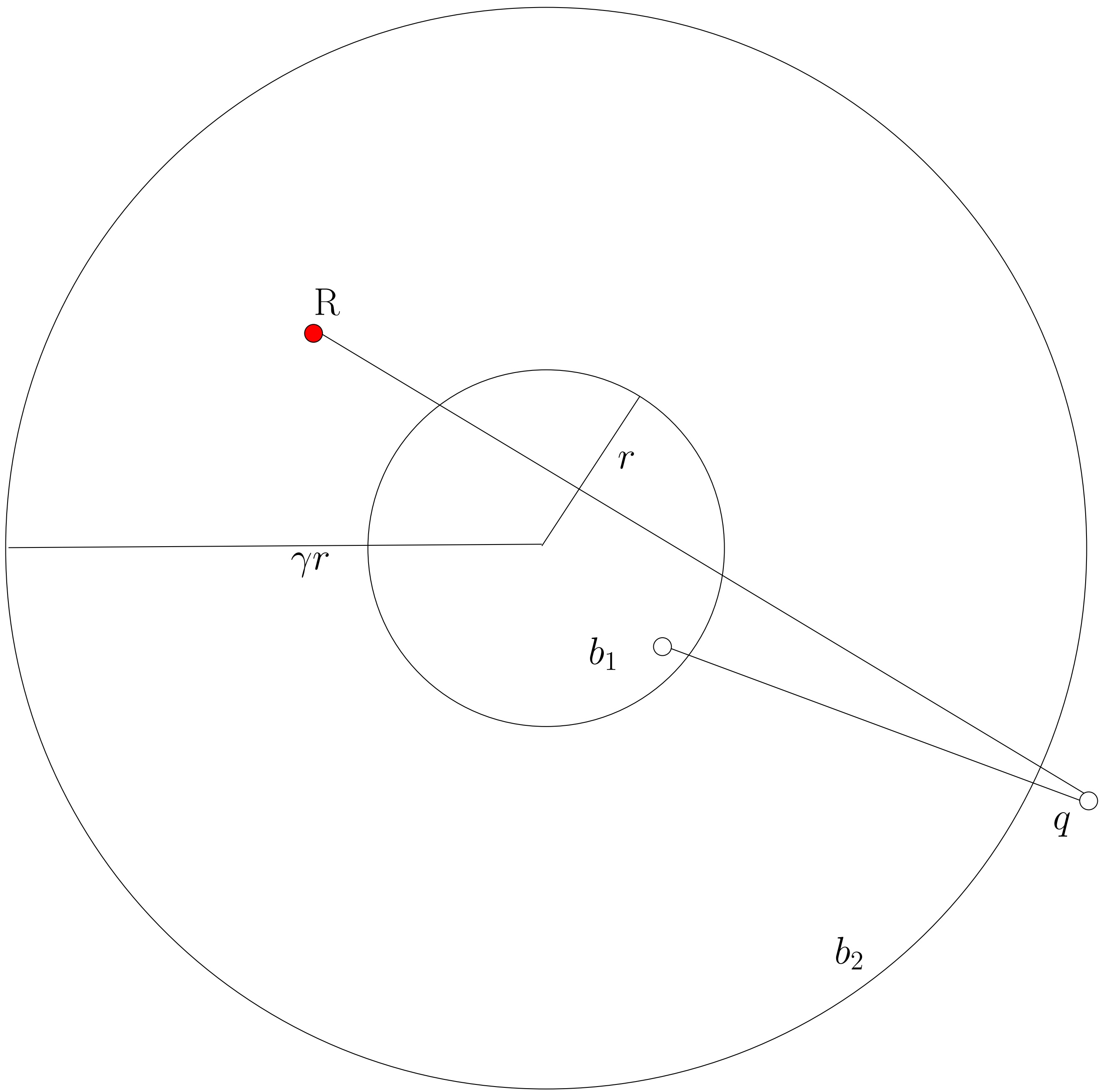
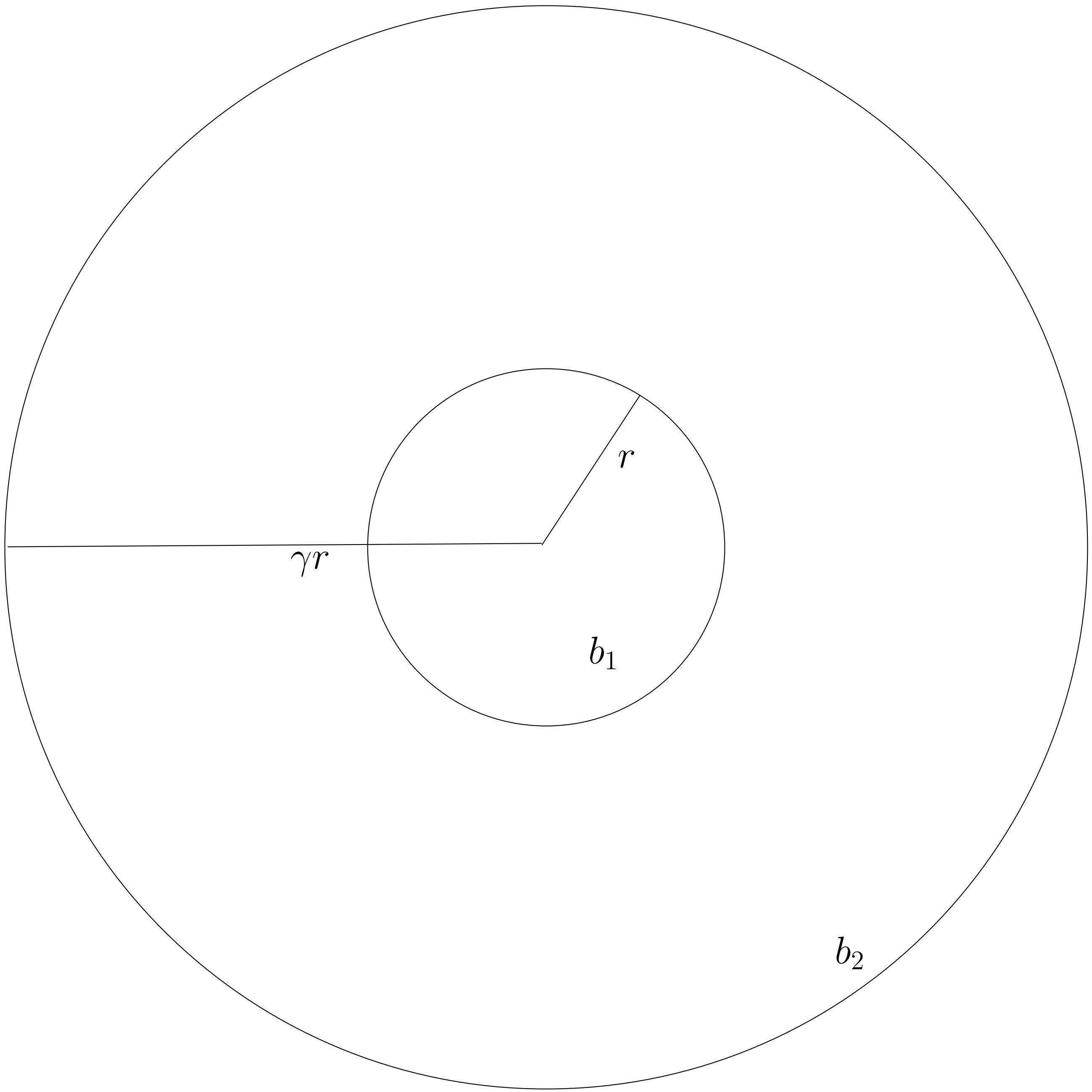


$$NN_q(R) \leq (1+\epsilon) NN_q(S \cap b_1)$$

Concentric Ball Lemma

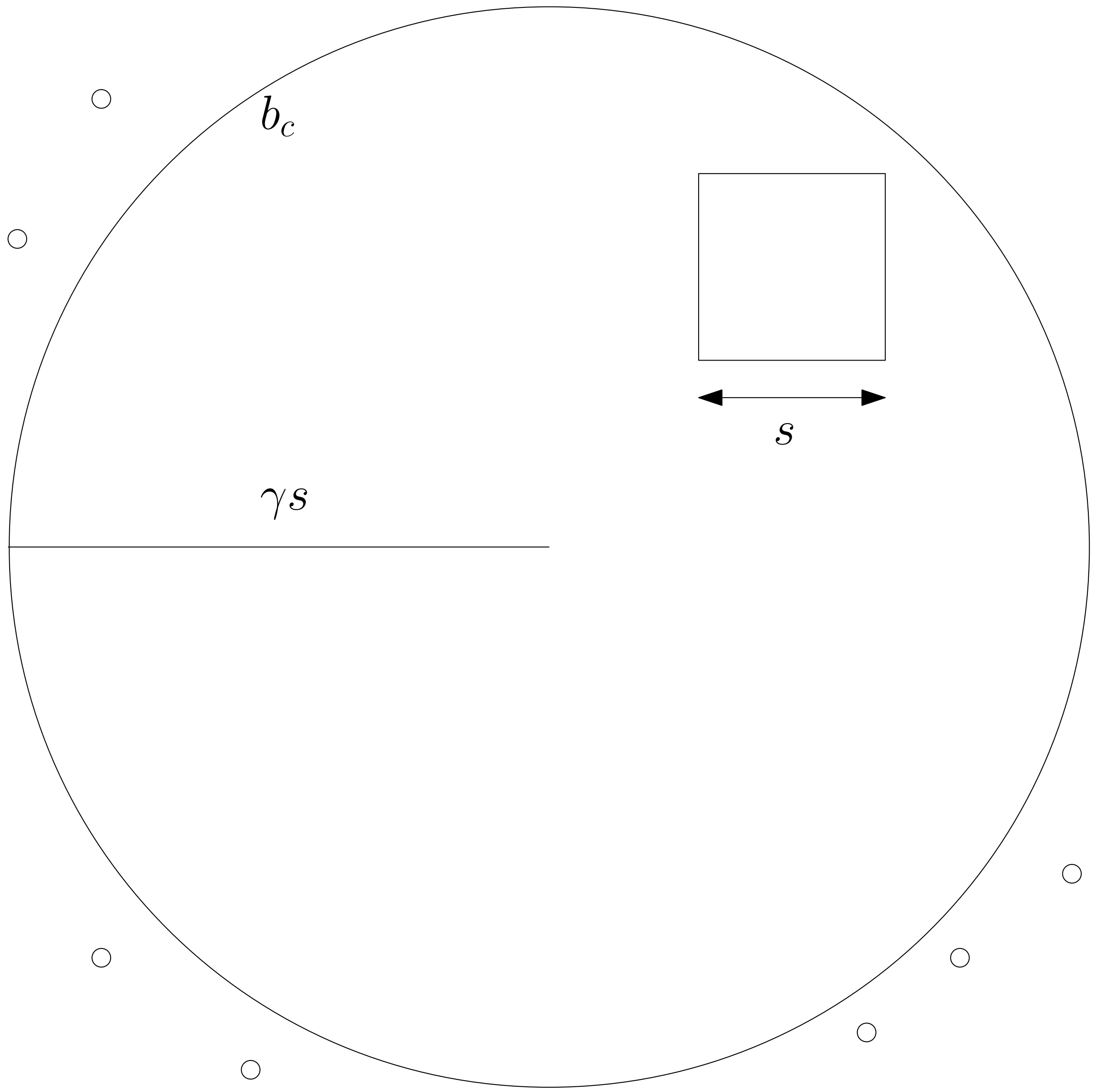


$$NN_q(R) \leq (1+\epsilon)NN_q(S \cap \overline{b_2})$$

$$NN_q(R) \leq (1+\epsilon)NN_q(S \cap b_1)$$

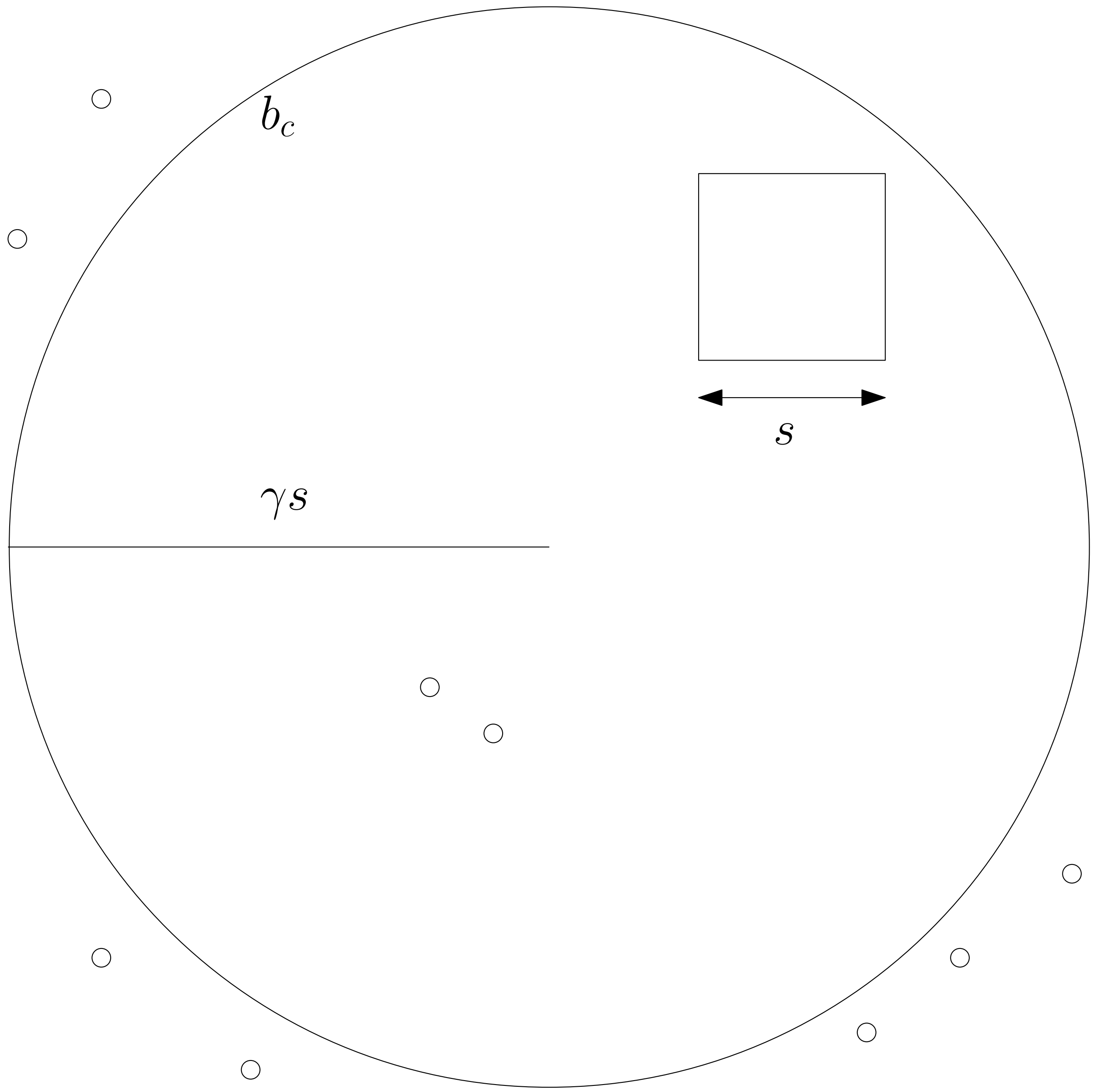
$$|R| = \left(1 + O\left(\frac{1}{\sqrt{\epsilon\gamma}}\right)\right)^{d-1}$$

Separation Lemma

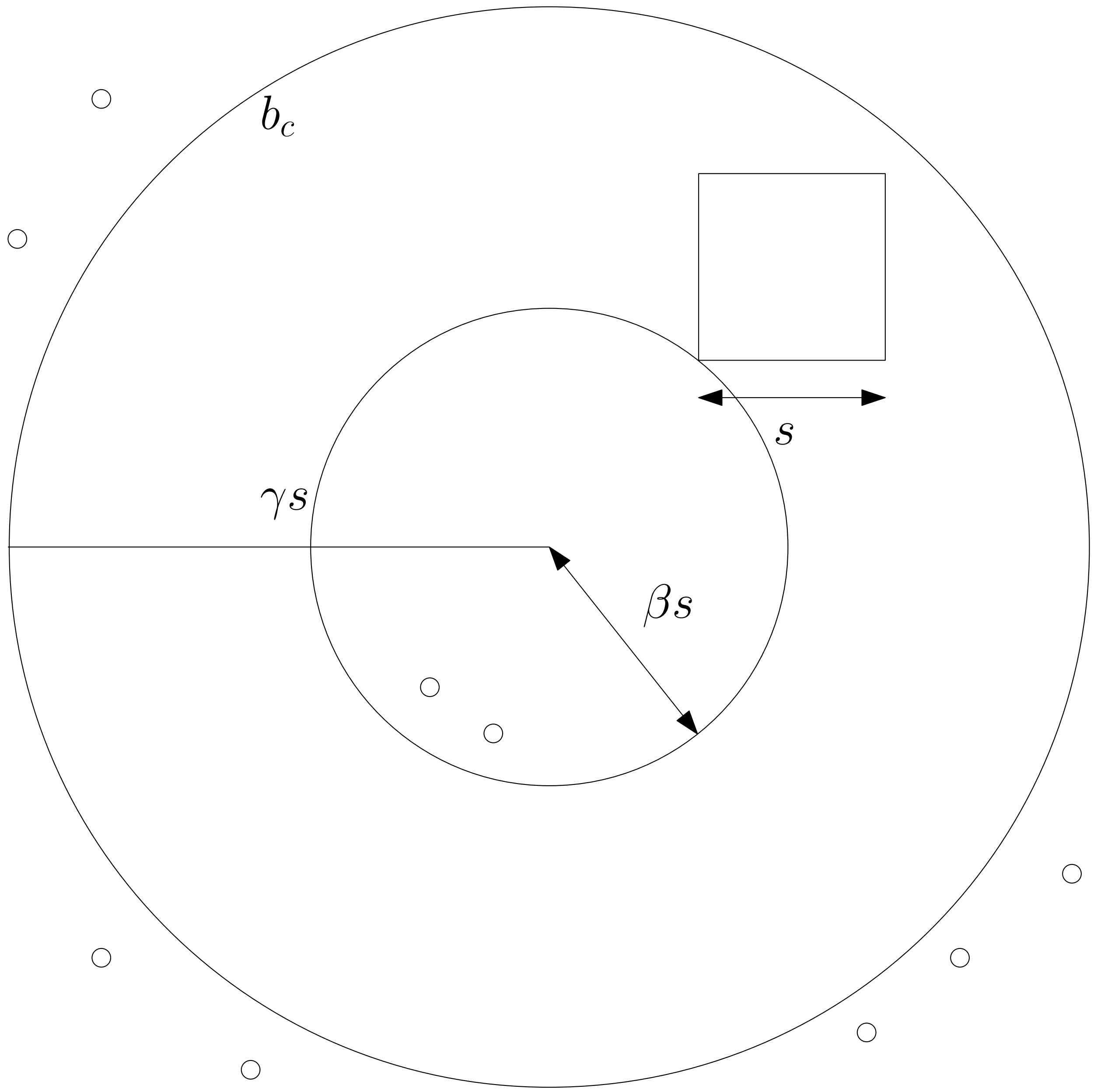


$|S \cap \gamma b_c| \leq 1$

Separation Lemma



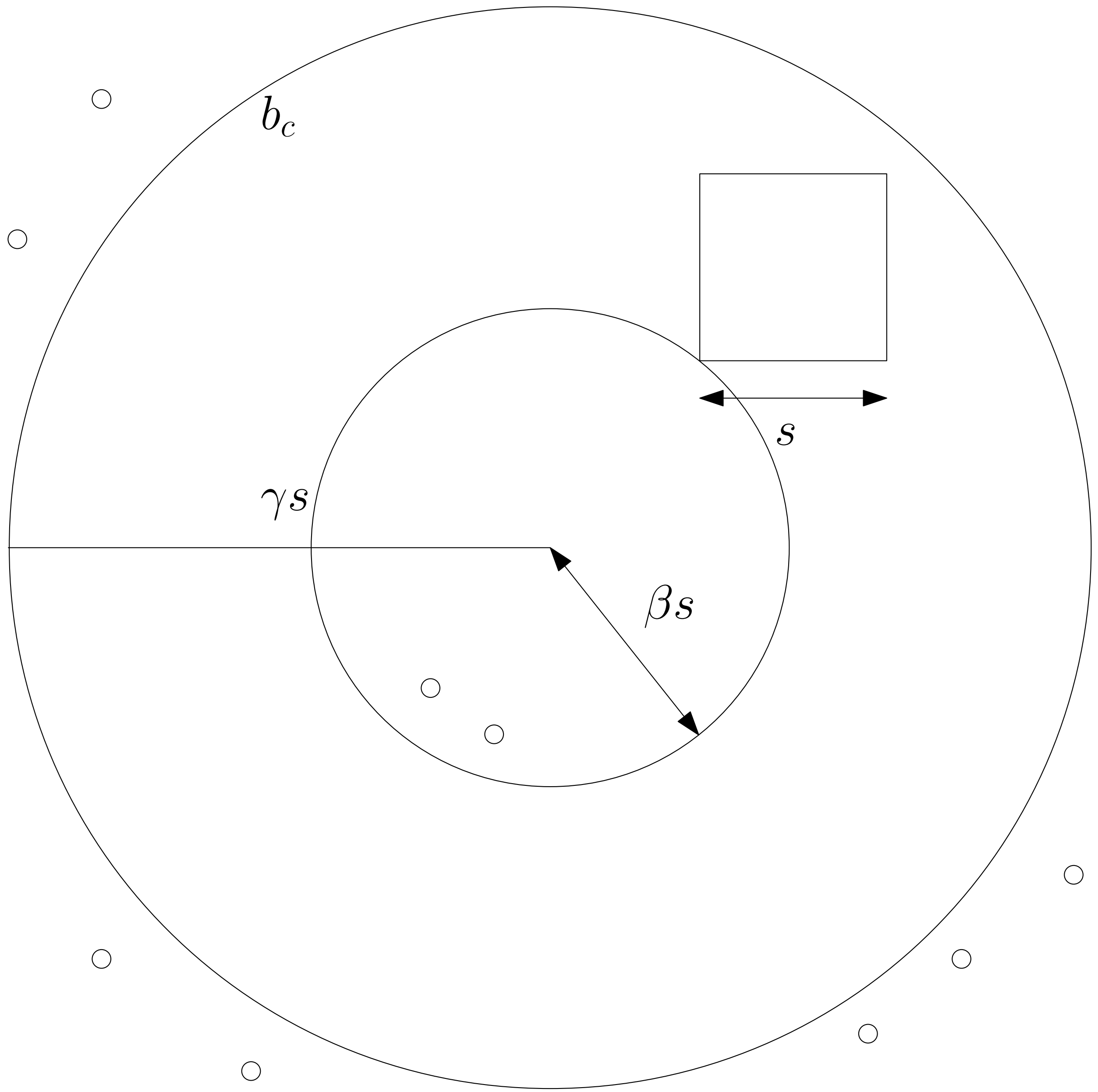
Separation Lemma



$$S \cap \gamma b_c \subseteq b'_c$$

$$\beta b'_c \cap c = \emptyset$$

Separation Lemma



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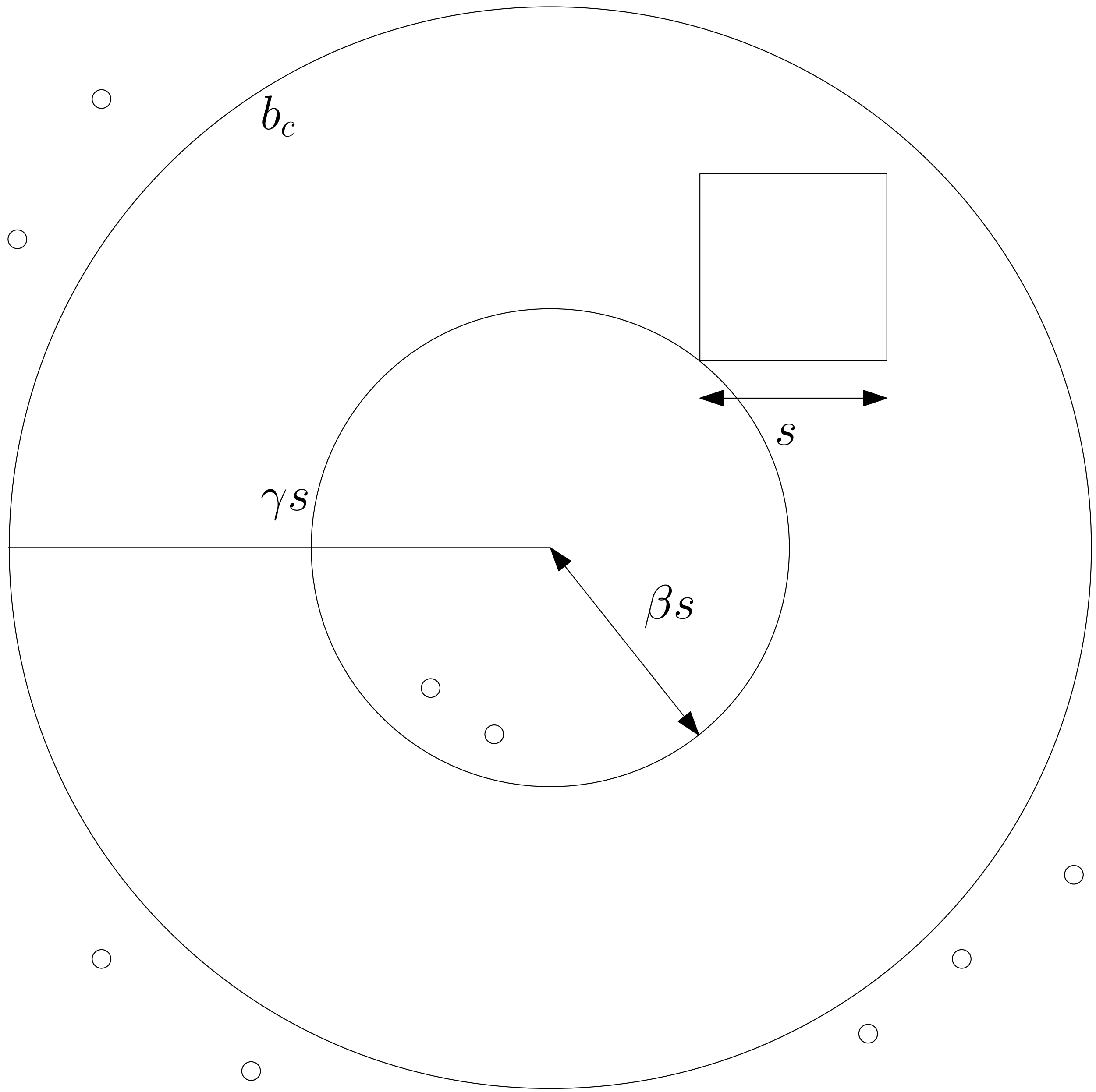
Why does this work?

Separation Lemma



$$S \cap \gamma b_c \subseteq b'_c$$

$$\beta b'_c \cap c = \emptyset$$

Why does this work?

If there was a point within γs not in βs then there would be a WSPD pair to force the cell to split

Choosing Representatives

- Choose R' consisting of $O(1/(\epsilon\gamma^{(d-1)/2}))$ points so that

$$NN_q(R') \leq (1+\epsilon)NN_q(S \cap \overline{\gamma b_c})$$

- If $|S \cap \gamma b_c| \leq 1 \Rightarrow R'' = S \cap \gamma b_c$
- Else R'' Consists of $O(1/(\epsilon\gamma)^{(d-1)/2})$ points such that

$$NN_q(R'') \leq (1+\epsilon)NN_q(S \cap b'_c) \leq (1+\epsilon)NN_q(S \cap \gamma b_c)$$

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$$\Rightarrow R = R' \cup R''$$

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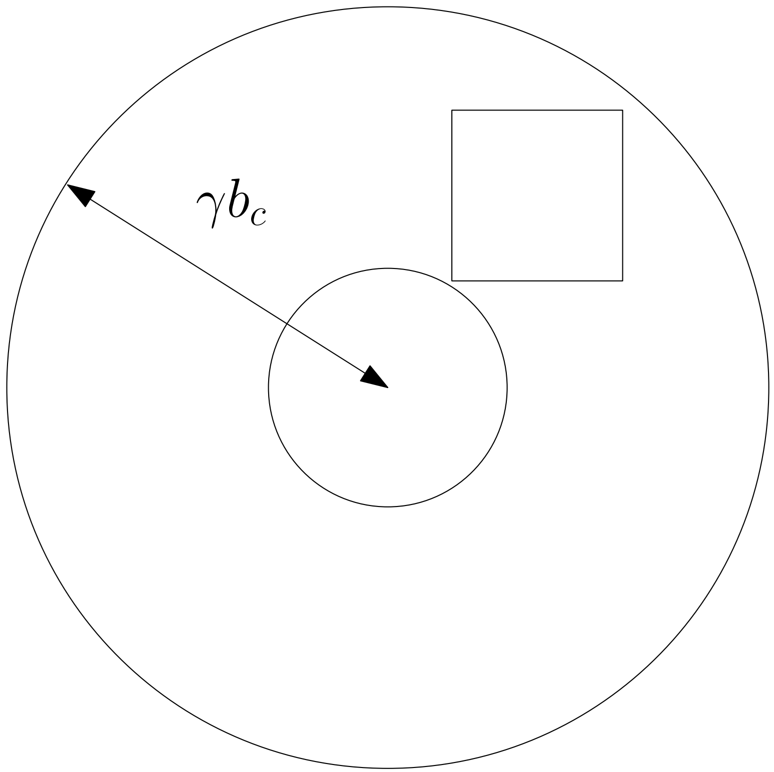

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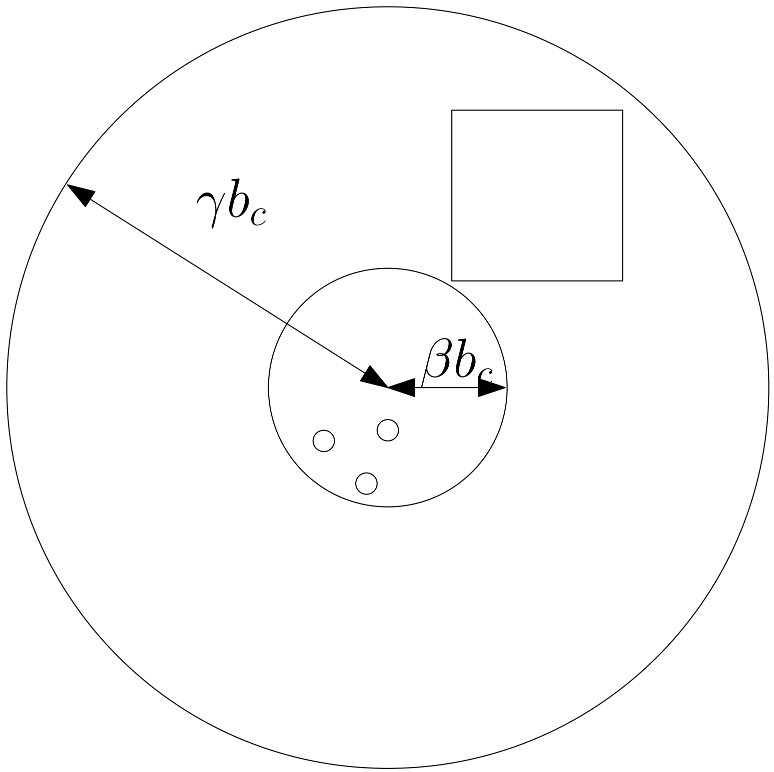

Choosing Representatives

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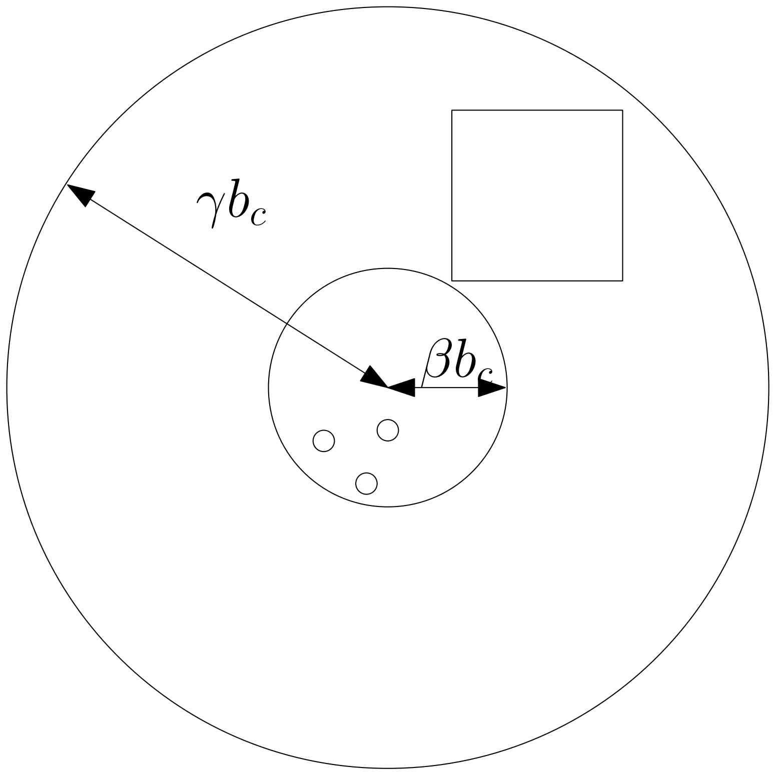
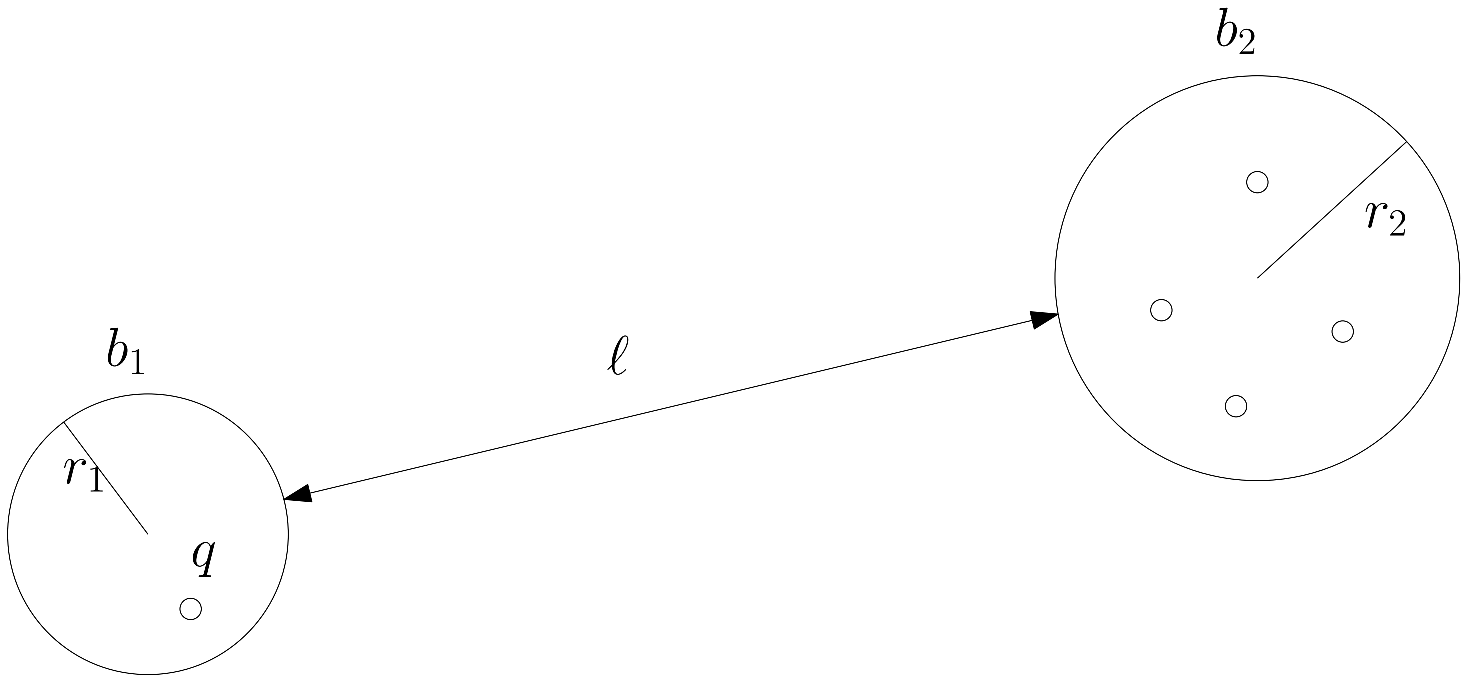


- $O(1/(\epsilon\gamma)^{(d-1)/2})$ number of representatives
- $O(n\gamma^d \log \gamma)$ cells

Disjoint Ball Lemma



$$NN_q(R) \leq (1+\epsilon)NN_q(S \cap b_2)$$

$$|R| = \left(1 + O\left(\frac{\sqrt{r_1 r_2}}{\ell\sqrt{\epsilon}}\right)\right)^{d-1}$$

Number of cells - number of representatives tradeoff

- Size of quadtree boxes can increase linearly with the WSPD distance

Before

$\Delta_b = r_b/(32\gamma d)$

Now

$\Delta_b = r_b^2/(256\ell\gamma d)$

Number of cells - number of representatives tradeoff

- Size of quadtree boxes can increase linearly with the WSPD distance

Before
$\Delta_b = r_b/(32\gamma d)$

Now
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Get fewer number of cells
$$O(\gamma^d \log \gamma) \rightarrow O(\gamma^d)$$

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$\Delta_b = r_b/(32\gamma d)$

Now

$\Delta_b = r_b^2/(256\ell\gamma d)$

Get fewer number of cells

$O(\gamma^d \log \gamma) \rightarrow O(\gamma^d)$

- Bounds

Size: $(O(1/(\epsilon\gamma)^{(d-1)/2}), \epsilon)$-approximate Voronoi diagram with $O(n\gamma^d)$ cells

Query Time: $O(\log(n\gamma) + 1/(\epsilon\gamma)^{(d-1)/2})$

Tradeoff Parameter γ: $2 \leq \gamma \leq 1/\epsilon$

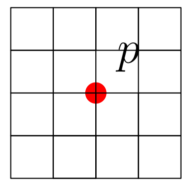
Summary

- Because any two points are well-separted in some pair choosing some close point is good enough
- Find representatives that are close to points
- Querying requires finding smallest quadtree cell
- With more representatives we need smaller separation and can use larger cells

Exponential Grid

- Definition

  For a given point p, the side length of a box is $r2^i$, where r is the distance from p



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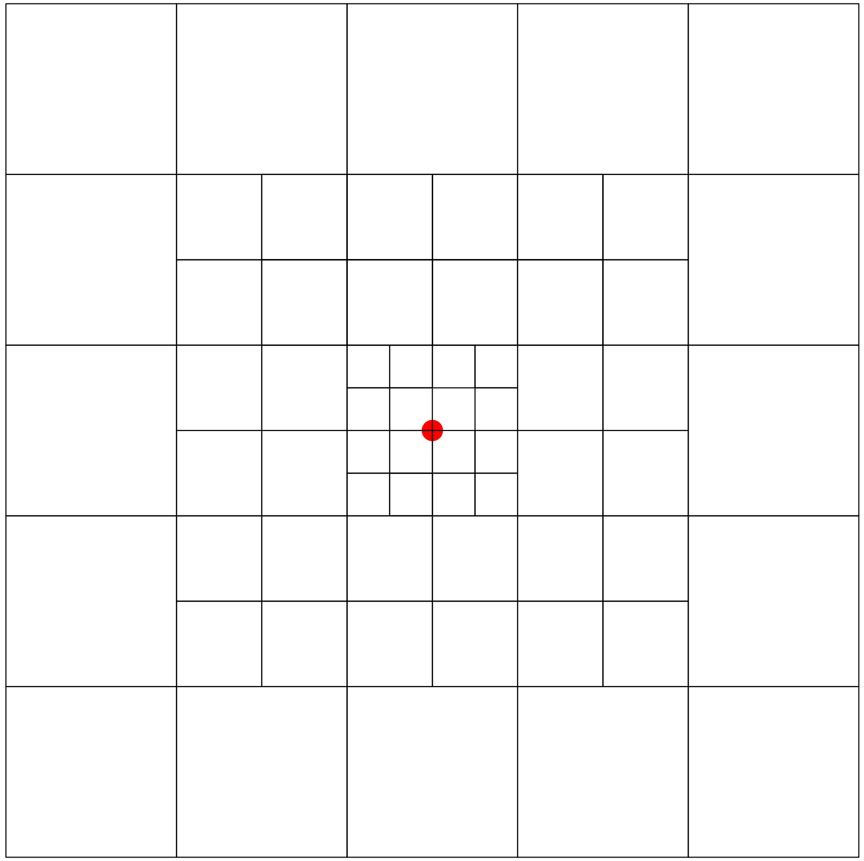

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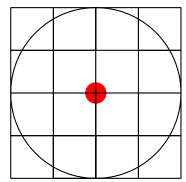

- Overlapping with a ball

Exponential Grid

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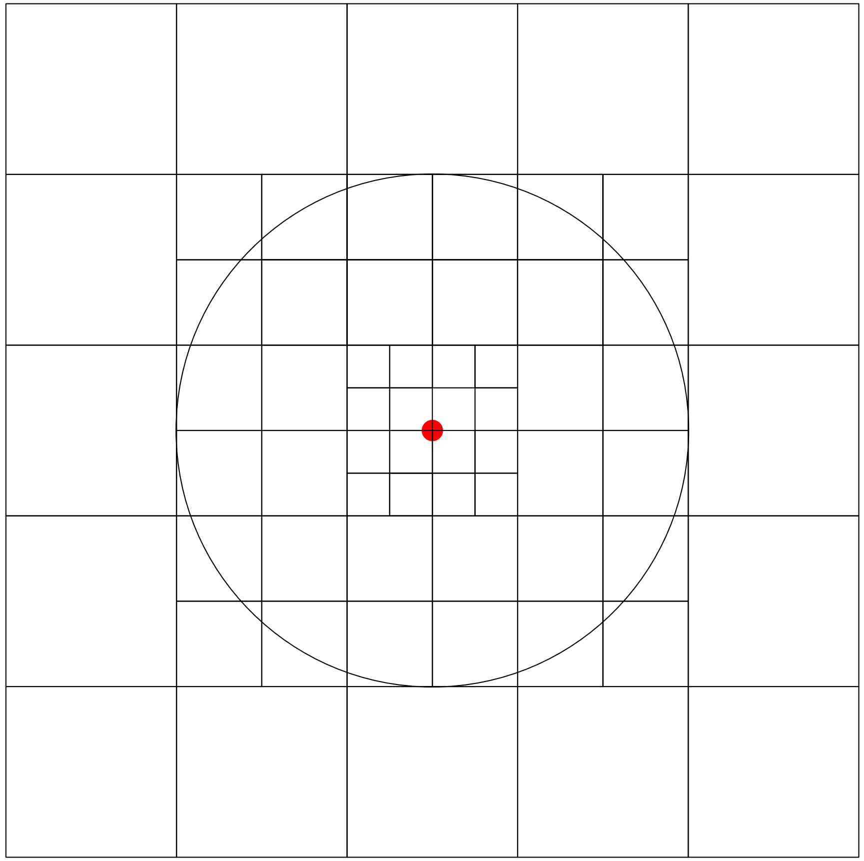
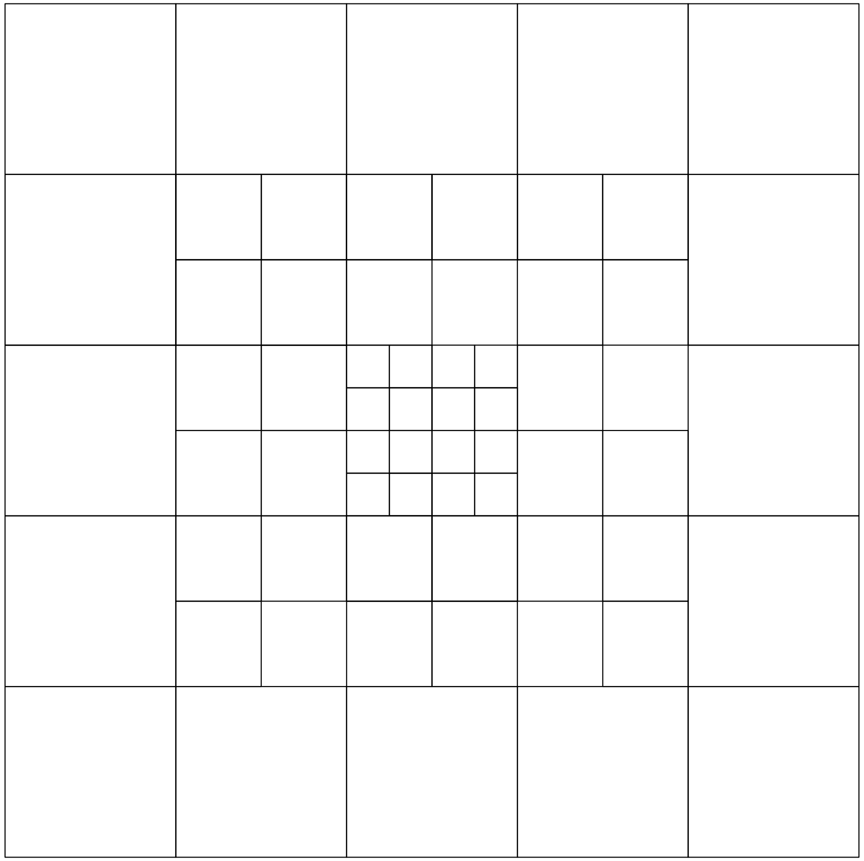
- Overlapping with a ball



Exponential Grid Arbitrary r



Exponential Grid Arbitrary r